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Stability problem in the dynamo

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ABSTRACT

It is shown that the saturated α-effect taken from the non-linear dynamo equations for the thin disc can still produce exponentially growing magnetic field in the case when this field does not feed back on the α. For negative dynamo number (stationary regime) stability is defined by the structure of the spectra of the linear problem for the positive dynamo numbers. The stability condition for the oscillatory solution (positive dynamo number) is also obtained and related to the phase shift of the original magnetic field, which produced saturated α and magnetic field in the kinematic regime. The results can be used to explain the similar effect observed in shell model simulations as well as in 3D dynamo models in the plane layer and sphere.

Key words: dynamo – instabilities – magnetic fields – turbulence.

1 INTRODUCTION

It is believed that the variety of the magnetic fields observed in astrophysics and experiments can be explained in terms of the dynamo theory (e.g. Hollerbach & Rüdiger 2004). The main idea is that kinetic energy of the conductive motions is transformed into the energy of the magnetic field. Magnetic field generation is a threshold phenomenon: it starts when the magnetic Reynolds number $R_{\rm m}$ reaches its critical value $R_{\rm m}^{\rm cr}$. After that, magnetic field grows exponentially up to the moment when it already can feed back on the flow. This influence does not result in the simple suppression of the motions and reduction of $R_{\rm m}$; rather, it results in changes to the spectra of the fields closely connected to constraints caused by conservation of the magnetic energy and helicity (Brandenburg & Subramanian 2005). The other important point is the effects of the phase shift and coherence of the physical fields before and after the onset of quenching discussed in Tilgner & Brandenburg (2008).

As a result, even after quenching the saturated velocity field is still large enough so that $R_{\rm m} \gg R_{\rm m}^{\rm cr}$. Moreover, the velocity field taken from the non-linear problem (when the exponential growth of the magnetic field stopped) can still generate exponentially growing magnetic field provided that the feedback of the magnetic field on the flow is omitted (kinematic dynamo regime; Tilgner 2008; Tilgner & Brandenburg 2008; Cattaneo & Tobias 2009; Schrinner et al. 2009). In other words, the problem of stability of the full dynamo equations including the induction equation, the Navier–Stokes equation with the Lorentz force, differs from the stability problem of the single induction equation with the given saturated velocity field taken from the full dynamo solution: stability of the first problem does not provide stability of the second one. The problem appears to be complex because some regimes close to Case 1 from the geodynamo benchmark (Tilgner 2008; Schrinner et al. 2009) are stable in contrast to solutions with periodical boundary conditions, and the influence of the boundary conditions can be important.

Here we consider the effect of such a type of stability on an example of the model of a galactic dynamo in the thin disc, as well as some applications to the dynamo in the sphere.

2 DYNAMO IN THE THIN DISC

One of the simplest galactic dynamo models is a one-dimensional model in the thin disc (Ruzmaikin, Shukurov & Sokoloff 1988):

$$\frac{\partial A}{\partial t} = \alpha B + A'',$$
$$\frac{\partial B}{\partial t} = -\mathcal{D}A' + B'', \qquad (1)$$

where A and B are azimuthal components of the vector potential and magnetic field, $\alpha(z)$ is a kinetic helicity, $\mathcal{D}$ is a dynamo number, which is a product of the amplitudes of the α- and ω-effects, and primes denote derivatives with respect to a cylindrical polar coordinate z. Equation (1) is solved in the interval $-1 \le z \le 1$ with the boundary conditions $B = 0$ and $A' = 0$ at $z = \pm 1$. We look for a solution of the form

$$(A,\ B) = {\rm e}^{\gamma t}(\mathcal{A}(z),\ \mathcal{B}(z)). \qquad (2)$$

Substituting (2) in (1) yields the following eigenvalue problem:

$$\gamma\mathcal{A} = \alpha\mathcal{B} + \mathcal{A}'',$$
$$\gamma\mathcal{B} = -\mathcal{D}\mathcal{A}' + \mathcal{B}'', \qquad (3)$$

where the constant γ is the growth rate. So as $\alpha(-z) = -\alpha(z)$ is an odd function of z, the generation equations have an important

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property: system (3) is invariant under the transformation $z \to -z$ when (Parker 1971)

$$\mathcal{A}(-z) = \mathcal{A}(z), \quad \mathcal{B}(-z) = -\mathcal{B}(z)$$

or

$$\mathcal{A}(-z) = -\mathcal{A}(z), \quad \mathcal{B}(-z) = \mathcal{B}(z). \tag{4}$$

Therefore, all solutions may be divided into two groups: odd on $\mathcal{B}(z)$, dipole (D) and even, quadrupole on $\mathcal{B}(z)$. Then we can replace $-1 \le z \le 1$ with the interval $0 \le z \le 1$ and the following boundary conditions at $z = 0$: $\mathcal{A}' = 0$, $\mathcal{B} = 0(D)$ and $\mathcal{A} = 0$, $\mathcal{B}' = 0(Q)$. Usually, $\alpha = \alpha_0$ with $\alpha_0(z) = \sin(\pi z)$ is used; see also Soward (1978) for $\alpha_0(z) = z$ dependence, more appropriate for analytical applications. Further, the sinusoidal form of $\alpha_0(z)$ is used.

System (3) has a growing solution, $\Re\gamma > 0$, when $|\mathcal{D}| > |\mathcal{D}^{\rm cr}|$. For $\mathcal{D} < 0$ the first exciting mode is quadrupole with $\mathcal{D}^{\rm cr} \approx -8$ and $\Im\gamma = 0$: the solution is non-oscillatory.[1] For $\mathcal{D} > 0$ the leading mode is oscillatory dipole, $\Im\gamma \neq 0$ with a higher threshold of generation: $\mathcal{D}^{\rm cr} \sim 200$. Putting non-linearity of the form

$$\alpha(z) = \frac{\alpha_0(z)}{1 + E_{\rm m}} \quad \text{for} \quad |B| \gg 1 \tag{5}$$

in (1), where $E_{\rm m} = (B^2 + A'^2)/2$ is a magnetic energy, gives stationary solutions for Q kind of symmetry and quasi-stationary solutions for D; see discussion of various forms of non-linearities in Beck et al. (1996). The property of the non-linear solution is mostly defined by the form of the first eigenfunction.

Now, in the spirit of Tilgner & Brandenburg (2008) and Cattaneo & Tobias (2009) we add to (1) equations for the new magnetic field $(\widehat{A}, \widehat{B})$ with the same α (5), which depends on (A, B) and does not depend on $(\widehat{A}, \widehat{B})$:

$$\begin{aligned}
\frac{\partial A}{\partial t} &= \alpha B + A'', \\
\frac{\partial B}{\partial t} &= -\mathcal{D}A' + B'', \\
\frac{\partial \widehat{A}}{\partial t} &= \alpha \widehat{B} + \widehat{A}'', \\
\frac{\partial \widehat{B}}{\partial t} &= -\mathcal{D}\widehat{A}' + \widehat{B}''.
\end{aligned} \tag{6}$$

Numerical simulations demonstrate that for the negative $\mathcal{D}$ both (A, B) and $(\widehat{A}, \widehat{B})$ are steady; however, the final magnitudes of $(\widehat{A}, \widehat{B})$ depend on the initial conditions for $(\widehat{A}, \widehat{B})$, see Fig. 1. The procedure was the following: equations (1) and (5) for (A, B) were integrated up to the moment $t = t_0$, then the full system (6) was simulated with the initial conditions for $(\widehat{A}, \widehat{B})$ in the form $(\widehat{A}, \widehat{B})|_{t=t_0} = (A, B)|_{t=t_0}(1 + C\varepsilon)$, where $\varepsilon \in [-0.5, 0.5]$ is a random variable and C is a constant. Both vectors (A, B) and $(\widehat{A}, \widehat{B})$ are stable in time; however, the final magnitude of $\widehat{E_{\rm m}}$ for $C \neq 0$ slightly depends on C. The presence of alignment of the fields (A, B) and $(\widehat{A}, \widehat{B})$ follows from linearity and homogeneity of the equations for $(\widehat{A}, \widehat{B})$, where $\alpha(z, E_{\rm m})$ is given. Later we consider the stability of $(\widehat{A}, \widehat{B})$ in more detail.

For $\mathcal{D} > 0$ the situation is different, resembling that of instability described in Cattaneo & Tobias (2009), Tilgner (2008), Tilgner & Brandenburg (2008) and Schrinner et al. (2009) for more sophisticated models: field $(\widehat{A}, \widehat{B})$ oscillates and starts to grow exponentially, see Fig. 2. Note that no regime in oscillations for $(\widehat{A}, \widehat{B})$ is observed. The other specific feature is the delay of $(\widehat{A}, \widehat{B})$ relative to (A, B): $\theta \approx -(\pi/3)$.

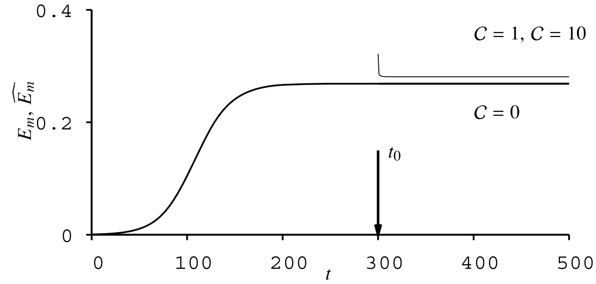

Figure 1. Evolution of magnetic energy $E_{\rm m}$ for $t < t_0$ governed by the system (1) and (5) for $\mathcal{D} = -10$. In the moment $t_0 = 300$ the new magnetic field $(\widehat{A}, \widehat{B})$ with the initial conditions defined by constant C is switched on, see (6) and (5). Plots for $E_{\rm m}$ and $\widehat{E_{\rm m}}$ (with $C = 0$) for $t > t_0$ coincide. All the solutions are stationary.

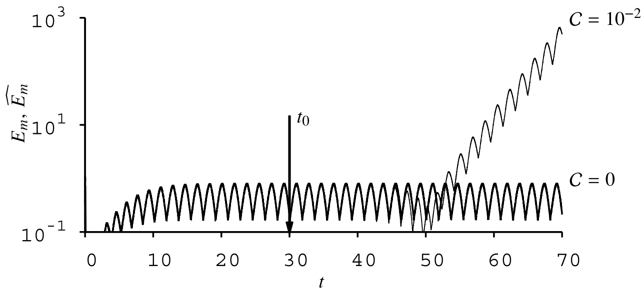

Figure 2. Evolution of magnetic energy $E_{\rm m}$ for $t < t_0$ governed by the system (1) and (5) for $\mathcal{D} = 300$. In the moment $t_0 = 30$ the new magnetic field $(\widehat{A}, \widehat{B})$ with the initial conditions defined by constant C is switched on, see (6) and (5). Plots for $E_{\rm m}$ and $\widehat{E_{\rm m}}$ (with $C = 0$) for $t > t_0$ coincide. For $C \neq 0$ after the intermediate regime, the phase shift θ between $E_{\rm m}$ and $\widehat{E_{\rm m}}$ increased and the exponential growth of $\widehat{E_{\rm m}}$ started.

If $E_{\rm m}$ in (5) is averaged over the space, so that α is steady, then the instability disappears. The question arises: does the instability depend on stationarity, or does it depend on something else?

It is known that for $\mathcal{D} < 0$ stability of the system (3) and (5), which has a stationary solution, is tightly bound to the behaviour of the linear solution of (3) for $\mathcal{D} > 0$ (Reshetnyak, Sokoloff & Shukurov 1992). Note that for the complex form of (3) it is equivalent to the solution of the conjugate problem.

Let $(\widetilde{A}, \widetilde{B}) = (\mathcal{A} + a, \mathcal{B} + b)$, where $(\mathcal{A}, \mathcal{B})$ is a solution of the non-linear problem and (a, b) is a perturbation with the same boundary conditions as for $(\mathcal{A}, \mathcal{B})$. Putting $(\widetilde{A}, \widetilde{B})$ in (3) with $\alpha \approx \alpha_0 + (\partial\alpha/\partial\mathcal{B})b$ yields equations for (a, b):[2]

$$\begin{aligned}
\gamma a &= \alpha^{\rm e} b + a'', \\
\gamma b &= -\mathcal{D}a' + b'',
\end{aligned} \tag{7}$$

where $\alpha^{\rm e} = \alpha + (\partial\alpha/\partial\mathcal{B})\mathcal{B}$ for $\alpha = [\alpha_0/(1 + \mathcal{B}^2)]$ is

$$\alpha^{\rm e} = \frac{1 - \mathcal{B}^2}{(1 + \mathcal{B}^2)^2}\alpha_0 \sim -\frac{\alpha_0}{\mathcal{B}^2} \quad \text{for} \quad |\mathcal{B}| \gg 1. \tag{8}$$

The behaviour of the $\alpha\omega$-dynamo (3) is defined by the sign of $\mathcal{D}\alpha$, and its change in the perturbed equations (7) is important. In other words, instead of non-linear equations (3) and (5) we come to the linear problem (3) with given $\alpha = \alpha(z, E_{\rm m})$ and effective dynamo

[1] For our Galaxy the usual estimate is $\mathcal{D} = -10$.

[2] It is usually supposed that in $\alpha\omega$-dynamo models $B \gg A'$.

number $\mathcal{D}^{e} = -(\mathcal{D}/\mathcal{B}^2)$. Then the stability of fields $(\widehat{A}, \widehat{B})$ for the negative $\mathcal{D}$ can be explained as follows. For the negative $\mathcal{D}$ solution $(\widehat{A}, \widehat{B})$ is finite and stable, because the threshold of generation $\mathcal{D}_{+}^{cr}$ for (3) is much larger than $\mathcal{D}^{e}$, $\mathcal{D}_{+}^{cr} \gg \mathcal{D}^{e}$. Field $(\widehat{A}, \widehat{B})$ is defined up to an arbitrary factor, which corresponds to alignment of the vectors (A, B) and $(\widehat{A}, \widehat{B})$. Note that $\mathcal{D}_{+}^{cr} \ll \mathcal{D}^{e}$ does not guarantee that $(\widehat{A}, \widehat{B})$ will grow exponentially due to non-linearity (5).

It is worthy of note that the non-linear solution of (1), (5) and (6), (5) demonstrates similar stationary behaviour even for $\mathcal{D} \sim -10^3$ in spite of the fact that $\mathcal{D}_{+}^{cr}$ for the quadrupole oscillatory mode for positive $\mathcal{D}$ is ∼200. The reason is that the dynamo system tends to the state of the strong magnetic field with $\mathcal{B} \sim \mathcal{D}^{1/2}$, so that $\alpha \sim 1/\mathcal{B}^2$, leaving $\mathcal{D}^{e}$ at the level of the first mode's threshold of generation.

For positive $\mathcal{D}$ (A, B), and therefore $\alpha(B)$, oscillate and one needs additional information on correlation of the waves. Here, instead of (8) we get $\alpha^{e} \sim -(\alpha_0 \widehat{B}/|B|^3)$. If the phase shift between B and $\widehat{B}$ is negligible, then the α-effect is saturated and time evolution of (A, B) and $(\widehat{A}, \widehat{B})$ is similar. However, simulations demonstrate in Fig. 2 that field $(\widehat{A}, \widehat{B})$ delays relative to (A, B). This is a typical situation, when parameter resonance takes place: α is modulated by signal with frequency $\Omega \sim 2\omega$, $\omega = \Im\gamma$, see Dawes & Proctor (2008) for details of the spatial resonance. This assumption is supported by the fact that instability disappears when in (5) a steady α, averaged in time, is used. Note that the situation is the same for the problem in the full volume $-1 \le z \le 1$, where instability depends on the form of quenching in the same way.

To demonstrate what happens, we consider how the delay θ of $(\widehat{A}, \widehat{B})$ relative to (A, B) changes the production of $\widehat{A}^2 + \widehat{B}^2$ near the threshold of generation $\mathcal{D}_{+}^{cr}$. We start from the linear analysis of the system in the form

$$\mathrm{i}\omega\widehat{A} = \alpha\widehat{B} - k^2\widehat{A},$$
$$\mathrm{i}\omega\widehat{B} = -\mathrm{i}\mathcal{D}_{+}^{cr}k\widehat{A} - k^2\widehat{B}. \quad (9)$$

From the condition of solvability for (9): $(k^2 + \mathrm{i}\omega)^2 = -\mathrm{i}\mathcal{D}_{+}^{cr}k\alpha_0$ with $\alpha = \alpha_0$, it follows that $\omega^2 = k^4 = 1$. The other prediction of the linear analysis is the phase shift φ between $\widehat{A}$ and $\widehat{B}$: $\varphi = \pm(\pi/4)$, which is twice as small as for the non-linear regime (Tilgner & Brandenburg 2008), so that for the non-linear regime the maximal $\widehat{A}$ is when $\widehat{B}$ is zero and quenching is absent.

Then putting in (6) $B = b\sin(x - t)$, $\widehat{A} = \sin(x - t + \varphi + \theta)$, $\widehat{B} = \sin(x - t + \theta)$ and $\alpha = 1/(1 + B^2)$, we obtain how the generation depends on θ. The equation for $\widehat{B}$ does not include the original field (A, B), so we consider only production of $\widehat{A}^2$. Then $\delta\widehat{A}(\varphi, \theta) = \alpha_0 \int_0^{2\pi} [\widehat{B}\widehat{A}/(1 + B^2)]\,\mathrm{d}t$. If $|\Pi| \gg 1$, where $\Pi = [\delta\widehat{A}(\varphi, \theta)/\delta\widehat{A}(\varphi, 0)]$, then $(\widehat{A}, \widehat{B})$ is unstable.

The exact equation for $\delta\widehat{A}$ is

$$\delta\widehat{A}(\varphi, \theta) = h_1 + h_2\tan(\varphi),$$
$$h_1 = \frac{\cos(\theta)^2(4 - 32^{1/2}) - 2(2^{1/2} - 1)}{2^{1/2}(2^{1/2} - 1)},$$
$$h_2 = \frac{\sin(2\theta)(32^{1/2} - 4)}{2^{3/2}(2^{1/2} - 1)}. \quad (10)$$

If $\theta = 0$ then $h_2 = 0$ and $\widehat{A}(\varphi, 0) = h_1 = 2^{1/2} - 4$. Then, for $\theta = (\pi/3)$, $h_1 = (2^{1/2} - 10)/4$, $h_2 = -(3^{1/2}/4)(2^{1/2} - 1)$, Π at $\varphi \to \pm(\pi/2)$ is singular and instability appears.

Summarizing the results for the steady and oscillatory dynamos, we have the following predictions for the stability of field $(\widehat{A}, \widehat{B})$. For $\mathcal{D} < 0$ (A, B) is steady and $(\widehat{A}, \widehat{B})$ is unstable when $|\mathcal{D}/(\mathcal{D}_{+}^{cr}B^2)| \gg 1$.

When (A, B) oscillates then $(\widehat{A}, \widehat{B})$ continues to oscillate with (A, B) increasing the phase shift between $(\widehat{A}, \widehat{B})$ and (A, B). Then instability caused by the parameter resonance may arise.

3 CONCLUSIONS

Here we argue that stability of the kinematic $\alpha\omega$-dynamo problem with the α-effect taken from the weakly non-linear regime near the threshold of generation can be predicted from knowledge of the threshold of generation of the linear problem with the opposite sign of the dynamo number. It appears that in spite of the fact that the magnetic field has already saturated α, it still can generate magnetic field if spectra of the linear problem are similar for dynamo number $\mathcal{D}$ with the opposite sign. So, as $\mathcal{D}$ depends on the product of the α and ω effects, a similar analysis can be performed with the ω-quenching, usually used in geodynamo models, see e.g. Soward (1978), as well as with the feedback of the magnetic field on diffusion. It is likely that for the more complex systems, the velocity field, taken from the saturated regime, with many exited modes will always generate magnetic field if the Lorentz force is omitted.

So as non-linearity (5) has quite a general form, we consider applications of these results to some other dynamo models. Linear analysis of the axisymmetrical $\alpha\omega$-equations gives the following (see Moffatt 1978 and references therein): for positive $\mathcal{D}$ (which is believed to be in the Earth) in the presence of the meridional velocity U_{p} the first exciting mode is dipole with $\Im\gamma = 0$. Reduction of U_{p} first leads to an oscillatory dipole solution (regime of frequent reversals; Braginskii 1964). The further reduction of U_{p} gives the quadrupole oscillatory regime with a larger value of $\mathcal{D}^{cr}$. For negative $\mathcal{D}$ and $U_{\mathrm{p}} \neq 0$ the first mode is quadrupole with $\Im\gamma = 0$. $U_{\mathrm{p}} \to 0$ gives a non-oscillatory dipole mode with decreased $\mathcal{D}^{cr}$ (see Meunier et al. 1997 for more details). In contrast to the dynamo in the disc the thresholds of generation for positive and negative $\mathcal{D}$ in the sphere are of the same order, and the situation with stability of the field $\widehat{\boldsymbol{B}}$ is uncertain, and can depend on the particular form of the α- and ω-effects. Anyway, stability of $\widehat{\boldsymbol{B}}$ for the steady regime is more likely.

In accordance with Cattaneo & Tobias (2009), shell models of turbulence demonstrate exponential growth of the magnetic field. This case, as well as 3D simulations of the turbulence in the box, which have the same instabilities, corresponds to the oscillatory regimes and using our predictions should be unstable.

In the case of the 3D dynamo in the sphere, simulations demonstrate different behaviour of $\widehat{\boldsymbol{B}}$ (Tilgner 2008; Schrinner et al. 2009). For small Rayleigh numbers, when the preferred solutions are dipole and oscillatory and close to the single mode structure in Case 1 in Christensen et al. (2001), $\widehat{\boldsymbol{B}}$ is finite. An increase of the Rossby number (Schrinner et al. 2009) leads to the turbulent state and $\widehat{\boldsymbol{B}}$ becomes unstable. It means that stability does not depend on the type of the boundary conditions as could be supposed from Tilgner (2008) and Tilgner & Brandenburg (2008), where vacuum and periodical boundary conditions for the magnetic field were used. Following our analysis, the more important factor for stability is the value of the time shift θ between the original magnetic field $\boldsymbol{B}$ and the passive field $\widehat{\boldsymbol{B}}$, which is still a free parameter in our model.

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