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THE SUBGRID PROBLEM OF THERMAL CONVECTION IN THE EARTH'S LIQUID CORE

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The problem of turbulent thermal convection in the Earth's liquid core is considered. Following assumptions on decreasing of the spatial scales due to rapid rotation, we propose a subgrid model of eddy diffusivity used in the large-scale model. This approach makes it possible to model realistic regimes with small Ekman and Rossby numbers ($\mathrm{E} \sim 10^{-14}$, $\mathrm{Ro} \sim 10^{-8}$) and with a sufficiently large Rayleigh number $\mathrm{Ra} \sim 10^{12}$. An estimate obtained for the averaged kinetic energy is comparable with observations. The model includes rotation of the solid core due to viscous torque.

1. Introduction. Convection in the Earth's liquid core caused by radioactive heating and compositional processes [1] is the subject of numerous researches, usually concerned with geomagnetic field generation as well. The last few decades are demonstrating a fascinating development in this area of research [2]. Based on the MHD large-scale equations, numerical simulations can reproduce different geomagnetic and geophysical phenomena: various properties of the geomagnetic field (e.g., its reversals and spectrum), eastward rotation of the Earth's inner core as well as the realistic ratio of kinetic and magnetic energies [3].

However, a wide range of spatial and time scales makes the direct numerical simulations (DNS) very cumbersome. The difficulty is caused by small values of transport coefficients: e.g., the kinematic viscosity of the liquid core is $\nu^{\mathrm{M}} = 10^{-6}\ \mathrm{m}^2\mathrm{s}^{-1}$ and the thermal diffusivity $\kappa^{\mathrm{M}} = 10^{-5}\ \mathrm{m}^2\mathrm{s}^{-1}$ (here the superscript M indicates the molecular values). This gives the following estimates of molecular Reynolds and Peclet numbers: $\mathrm{Re}^{\mathrm{M}} = \dfrac{V_{\mathrm{wd}}L}{\nu^{\mathrm{M}}} \sim 10^9$ and $\mathrm{Pe}^{\mathrm{M}} = \dfrac{V_{\mathrm{wd}}L}{\kappa^{\mathrm{M}}} \sim 10^8$, where $V_{\mathrm{wd}} = 0.2^\circ\ \mathrm{year}^{-1}$ is the west drift velocity and $L = 3\cdot 10^6$ m is the liquid core scale [4], which corresponds to the regime of highly developed turbulence. In the case of Kolmogorov's turbulence 3D DNS, simulations require $\sim \mathrm{Re}^{9/4} = 10^{20}$ grid nodes [5]. Attempts to use the exact values of these parameters on coarse grids lead to numerical instabilities. The first intuitive models in geodynamo theory that suppress instabilities at small scales (e.g., the model of hyperdiffusivity [3]) gave rise to new questions concerned with interpretation of the results obtained [6]. The more consistent way is an application of semiempirical turbulence models [7]. Usually, these models are based on assumptions on the cascade transfer similar to Kolmogorov's, which gives descriptions of the average effect of small-scale field fluctuations on the large-scale flow in terms of eddy diffusivity. The recent studies on the subgrid [8] and more complicated models [9, 10] for the thermal convection and dynamo problems in rotating spheres revealed the principal possibility of describing the small-scale fluctuations in turbulence with desired Reynolds and Peclet numbers, much like Kolmogorov's model.

These models work up to the regime of moderate rotation speed. The further increase in the Coriolis force can reduce the total kinetic energy and even suppress convection at all. From linear analysis it follows that the critical Rayleigh number depends on the Ekman number like $\mathrm{Ra}^{\mathrm{cr}} \sim \mathrm{E}^{-1/3}$ [11]. Even though the molecular estimate of the Rayleigh number gives huge numbers $\mathrm{Ra}^{\mathrm{M}} \sim 10^{14}$ [12, 13], this value is only $5\cdot 10^2$ times larger than the critical value $\mathrm{Ra}^{\mathrm{cr}}$ [2]. Due to rapid rotation of the Earth, the situation in the liquid core is more complicated and assumptions on similarity of the field spectral characteristics must be checked very carefully. We show that the direct applications of the traditional turbulence models based on the mixing-length assumptions lead to results that differ from the observations by orders of magnitude. The cause of such a disagreement lies in the daily rapid rotation of the Earth, which gives rise to new characteristic spatial scales in the core [14]. As a result, the energy distribution in the spectrum changes, which leads to difficulties for the application of Prandtl–Kolmogorov's approach to eddy diffusivity estimation. Convection at these new scales plays a crucial role in the energy balance of the whole system and changes the estimate of total energy by orders

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of magnitude. Even a simple account of these effects leads to substantial change in the rate of energy dissipation and thus to a better agreement of LSS models with observations.

In Section 2 we introduce the large-scale equations of thermal convection and consider Prandtl–Kolmogorov's assumptions on eddy diffusion. In Section 3 we recall the foundations of convection in a rapidly rotating body and obtain an estimate for subgrid diffusion. Afterwards, this estimate is used in the large-scale model, Section 4. Our discussion of results is in Section 5.

2. The large-scale equations. The problem of thermal convection in the Earth's core can be reduced to that in a spherical shell. Let the surface of a sphere of radius r_0 (in the spherical system of coordinates (r, θ, φ)) rotate with an angular velocity Ω about the z-axis. This sphere contains a concentric solid inner sphere of radius r_i; the outer spherical layer ($r_i < r < r_0$) is filled with an incompressible liquid ($\mathbf{V} = 0$). The solid inner sphere may rotate freely about the z-axis due to viscous torque. Convection in the outer sphere is described in the Boussinesq approximation by the Navier–Stokes equation and by the heat-flux equation. Choosing $L = r_0$ as unit of length, we can measure the velocity $\mathbf{V}$, the time t, and the pressure p in units of κ^{M}/L, L^2/κ^{M}, and $2\Omega\rho\kappa^{\mathrm{M}}$, respectively. Then, the governing equations can be written in the form

$$\mathrm{Ro}^{\mathrm{M}}\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right] = -\nabla p - \mathbf{1}_z \times \mathbf{V} + \mathrm{Ra}^{\mathrm{M}}\, T r \mathbf{1}_z + \mathrm{E}^{\mathrm{M}}\,\nabla\cdot \overset{\leftrightarrow}{\mathbf{S}}, \qquad (1)$$

$$\frac{\partial T}{\partial t} + (\mathbf{V}\cdot\nabla)\,(T + T_0) = \nabla\cdot(\nabla T), \qquad (2)$$

where $\mathbf{1}_z$ is the unit vector in z-direction, $\overset{\leftrightarrow}{\mathbf{S}}$ is the strain tensor, and T is the temperature fluctuations relative to the imposed profile $T_0 = \dfrac{r_{\mathrm{i}}/r - 1}{1 - r_{\mathrm{i}}}$. The molecular Rossby Ro^{M}, Ekman E^{M}, and Rayleigh Ra^{M} numbers appear in these equations:

$$\mathrm{Ro}^{\mathrm{M}} = \frac{\kappa^{\mathrm{M}}}{2\Omega L^2}, \quad \mathrm{E}^{\mathrm{M}} = \frac{\nu^{\mathrm{M}}}{2\Omega L^2}, \quad \mathrm{Ra}^{\mathrm{M}} = \frac{\alpha g_0 \delta T L}{2\Omega\kappa^{\mathrm{M}}}.$$

Here α is the thermal expansion coefficient, g is the gravity acceleration, and δT is a temperature unit ($\delta T \sim 10^{-4}$ K, see [2]). It should be mentioned that the Rayleigh number for nonrotating bodies is usually given in the form $\widetilde{\mathrm{Ra}}^{\mathrm{M}} = \alpha g \delta T L^3/\kappa^{\mathrm{M}}\nu^{\mathrm{M}}$ and that $\mathrm{Ra}^{\mathrm{M}} = \mathrm{E}^{\mathrm{M}}\,\widetilde{\mathrm{Ra}}^{\mathrm{M}}$.

The nondimensionalized momentum equation for the angular velocity ω of the inner sphere ($0 < r < r_{\mathrm{i}}$) is of the form

$$\mathrm{Ro}^{\mathrm{M}}\, I\,\frac{\partial \omega}{\partial t} = r_{\mathrm{i}}\ \mathrm{E}^{\mathrm{M}} \oint_{\mathcal{S}} S_{\varphi r}\big|_{r=r_{\mathrm{i}}} \sin\vartheta\, d\mathcal{S}, \qquad (3)$$

where I is the moment of inertia of the inner sphere $\mathcal{S}$ and $S_{\varphi r}$ is the strain tensor component in the spherical system of coordinates [2]. Equations (1), (2), and (3) are accompanied by the nonpenetrating and no-slip boundary conditions for the velocity $\mathbf{V}$ and for zero temperature fluctuations at the shell boundaries.

The system (1), (2), (3) was successfully studied in the regimes of laminar convection with the use of different numerical approaches [2, 16]. However, these regimes are still very far from the desired estimates for the Earth's liquid core $\mathrm{Ro}^{\mathrm{M}} = 10^{-8}$, $\mathrm{E}^{\mathrm{M}} = 10^{-14}$, and $\mathrm{Ra}^{\mathrm{M}} = 10^{14}$ [2]. Attempts to approach to these parameters using DNS caused numerical instabilities and required the application of turbulence models [8]. However, even the direct usage of the known turbulence models is not trivial.

To support this point, we offer a simple estimate of eddy diffusivity on the basis of the most popular mixing-length assumption. Following Prandtl–Kolmogorov's hypothesis, the eddy diffusion at scale l can be estimated as $\nu^{\mathrm{T}} = (\varepsilon l^4)^{1/3}$, where $\varepsilon = v^3/l$ is the energy dissipation rate and v is the velocity at scale l. Even the upper-bound estimate based on the main scale $l = L$ and the west drift velocity $V = 3\cdot 10^{-3}$ m s^{-1} gives $\nu^{\mathrm{T}} = 2\cdot 10^3$ m^{2}s^{-1} for an Ekman number of order $\mathrm{E}^{\mathrm{T}} = 2\cdot 10^{-5}$. The more realistic estimate with $V_l = \delta V \sim V\left(\dfrac{l}{L}\right)^{1/3}$ and a usual grid scale of $l \sim 3\cdot 10^{-2}L$ yields $\nu^{\mathrm{T}} = 15$ and $\mathrm{E}^{\mathrm{T}} = 10^{-7}$. On the other hand, this estimate of E^{T} would require resolution of about $N_\varphi \sim 2\cdot 10^2$ cylindrical columns in the flow [11], which necessitates the use of the most powerful modern computers. All this means that the above estimate of ν^{T} will not provide the smooth field behavior assumed in Kolmogorov's turbulence when $\mathrm{E}^{\mathrm{T}} \geqslant 1$. Thus, the traditional methods underestimate the eddy diffusion ν^{T}.

Such a situation corresponds to the case for which the classical ideas on the direct cascade of energy from the main scale L to the dissipative scale are violated and additional information on ε at dissipative scale is needed. In Section 3 we show that in the case of a rapidly rotating body the energy in the spectrum is shifted to small scales unresolved by DNS even at the onset of convection. This is the reason why any attempt to estimate ν^{T} in the turbulent regime at scales compared to the grid resolution lead to the not selfconsistent behavior of the turbulent model.

The way out of such difficulties is to formulate proper assumptions on the spectral properties of the solution in the range of high wavenumbers.

3. Predictions of asymptotic analysis. The origin of the problem can be seen from the analysis of linearized system (1), (2) at the onset of convection in the limit of small Rossby and Ekman numbers. As shown in [11] (see also the recent paper [17]), at the onset of convection the flow structure tends to develop columns along z-direction such that $\partial/\partial\varphi \sim O(\mathrm{E}^{-1/3})$, $\partial/\partial s \sim O(\mathrm{E}^{-1/6})$, and $\partial/\partial z \sim O(1)$ when $\mathrm{E} = \mathrm{Ro} \to 0$. Linearization of system (1), (2) leads to the balance of Archimedean and viscous terms in the Navier–Stokes equation: $\mathrm{Ra}\, T \sim \mathrm{E}^{-1/3} V$. The balance of convective and viscous terms in the heat-flux equation gives $V \sim \mathrm{E}^{-2/3} T$ from which follows the estimate of the critical Rayleigh number $\mathrm{Ra}^{\mathrm{cr}} \sim \mathrm{E}^{-1/3}$. (For convenience we omitted superscript M.) Thus, at the onset of convection for system (1), (2), the flow is anisotropic with the smallest scale $l_E \sim \mathrm{E}^{1/3} L$ defined by the balance of Coriolis and viscous forces. Note that the scale $l_E \sim 10^{-5}$ is beyond the level of DNS. If this asymptotic behavior is valid, the critical Rayleigh number in the Earth's core is $\mathrm{Ra}^{\mathrm{cr}} \sim 10^5$ [2]. As we show below, the predicted column-like form of flow is very important for estimates of subgrid dissipation in the liquid core.

The main assumption is that even in the turbulent regime believed to be in the Earth's liquid core, the flow tends to elongated structures with the smallest scale l_E predicted by linear analysis. It is from the scale l_E, the ideas of the direct cascade of energy are applicable. To simplify the problem, we estimate the isotropic eddy diffusion on the basis of the scale (l_E). In particular, instead of the estimate of velocity gradient at the subgrid scale l: $V' \sim \delta V/l$, we use $V' \sim \mathrm{E}^{-1/3}\,\delta V$, where $\delta V \sim 0.3V$ is the average variation of velocity at scale l. In this case the estimate of eddy diffusion gives $\nu^{\mathrm{T}} \sim l^2 V' \approx 5\cdot 10^4\ m^2\, s^{-1}$ and $\mathrm{E}^{\mathrm{T}} \sim 5\cdot 10^{-4}$. This estimate of turbulent Ekman number corresponds to $N_\varphi \sim 10$ columns that can be resolved in the large-scale models with desired accuracy. To demonstrate these arguments, we propose simulations of system (1), (2), (3) with the given eddy diffusion ν^{T} estimated above.

4. A turbulent model. Numerical results. Equations (1), (2), and (3) are solved using the control-volume method (Simple algorithm) [18] on the staggered grid $(n_{\mathrm{r}},\, n_\theta,\, n_\varphi) = (45, 45, 64)$. This method is based on a finite-difference approximation and demonstrates very high numerical stability for the regimes with intensive convection[3] . For ease of calculation, we renormalize these equations, using turbulent diffusion units, so that instead of κ^{M} the value $\widehat{\kappa} = 1\ m^2\, s^{-1}$ is used. Then, the dimensionless parameters become $\mathrm{Ro}^{\mathrm{T}} = \dfrac{\widehat{\kappa}}{2\Omega L^2} = 4\cdot 10^{-2}$ and $\mathrm{E}^{\mathrm{T}} = \dfrac{\nu^{\mathrm{T}}}{2\Omega L^2} = 10^{-3}$. We consider the three regimes with turbulent Rayleigh numbers $\mathrm{Ra}^{\mathrm{T}} = \dfrac{\alpha g_0 \delta T L}{2\Omega\widehat{\kappa}} = 10^6,\ 10^7$, and 10^8 (see the time evolution of kinetic energy E_{K} in Figure 1). The corresponding Reynolds numbers averaged over the shell volume $\mathrm{Re}^{\mathrm{M}} = \dfrac{\widehat{\kappa}}{\nu^{\mathrm{M}}}\sqrt{2E_{\mathrm{K}}}$ are $3\cdot 10^9$, $6\cdot 10^9$, and $2\cdot 10^{10}$ (c.f. with the molecular Reynolds number for the Earth's core based on the west drift velocity $\mathrm{Re}^{\mathrm{Earth}} \sim 10^9$).

Some characteristic snapshots of large-scale velocity r-, θ-, and φ-components are presented in Figure 2. The observed curls in r- and φ-projections correspond to the columns parallel to the z-axis. These columns may drift in φ-direction. In its turn, the nonzero viscous gradient $\nu\dfrac{\partial}{\partial r}\left(\dfrac{V_\varphi}{r}\right)_{r=r_{\mathrm{i}}}$ causes rotation of the inner core (see the evolution of the inner core angular velocity ω in Figure 1). Here the positive value of ω corresponds to the eastward direction known to occur in the Earth [20]. We emphasize that these maps are a result of averaging of small-scale ($l_E \sim O({\mathrm{E}^{\mathrm{M}}}^{1/3}) = 10^{-5}$) structures. So far, the micro-scale Reynolds number $\mathrm{r_e}$ at scale l_E is still larger than unity and the inertial spectrum for the scales smaller than l_E exists. An estimate of $\mathrm{r_e} = \dfrac{vl}{\nu^{\mathrm{M}}}$ with $l = {\mathrm{E}^{\mathrm{M}}}^{1/3}$ and $v = 0.1V$ gives $\mathrm{r_e} \sim 10^3$. This spectrum has two parts with the transition point defined by the balance of inertial and Coriolis terms: $l_\Omega \sim \mathrm{Ro}\, v$. The turbulence in the range $l_E \leqslant l_\Omega$ is influenced by rotation; the kinetic energy spectrum is $E_l \sim l^2$ [21]. For the scales smaller than l_Ω up to the dissipative scale $l_d = \mathrm{Re}^{-3/4}$, Kolmogorov's spectrum $E_l \sim l^{5/3}$ reappears.

Summarizing the above results, we conclude that based on the realistic values of Rossby and Rayleigh numbers and on assumptions on the flow spectrum in the liquid core we obtained a value of kinetic energy E_{K} comparable to observations. Keeping in mind that the velocity field and the eddy diffusion are associated in

[3] See [19] for some special considerations of the control-volume method for the full dynamo problem in a sphere.

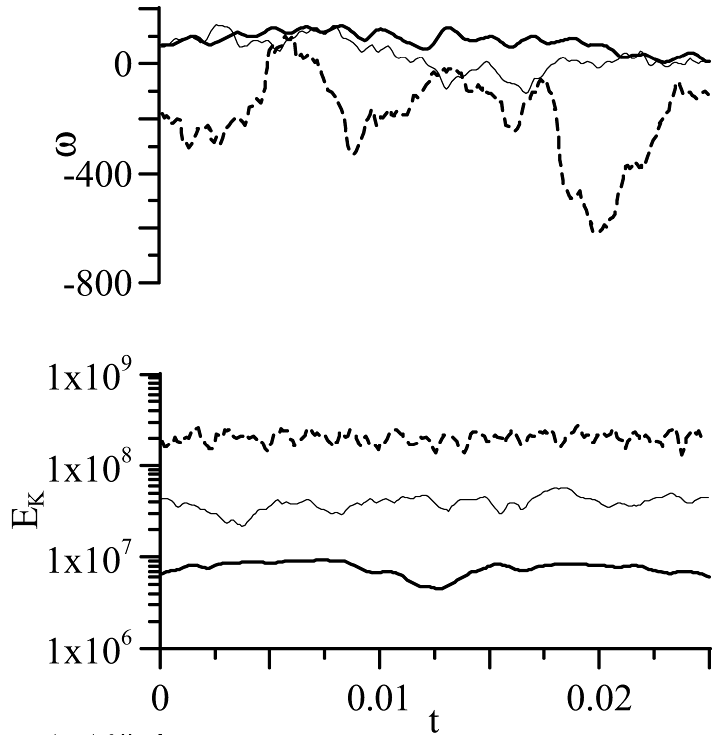

Fig. 1. Evolution of the liquid core angular velocity ω and the kinetic energy $E_{\rm K}$ averaged over the volume for $\rm Ro = 4 \cdot 10^{-7}$ and $\rm E = 10^{-14}$; (1): $\rm Ra^T = 10^6$ — thick line, (2): $\rm Ra^T = 10^7$ — thin line, (3): $\rm Ra^T = 10^8$ — dashed line

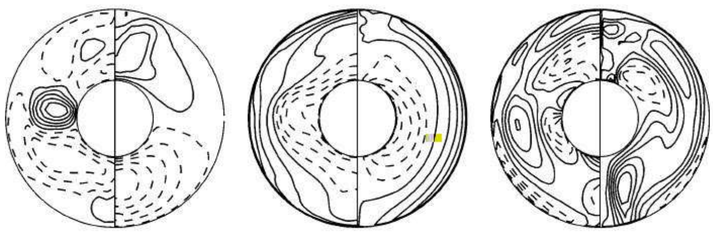

Fig. 2. The snapshots of the velocity field components v_r, v_ϑ, and v_φ (from left to right) for the equatorial sections (the left half-plane): $(-700, 1200)$, $(-4700, 4100)$, $(-1200, 1800)$ and meridional sections for the axisymmetric parts of the fields (the right half-plane): $(-2400, 1800)$, $(-3800, 3100)$, $(-400, 1000)$. Here the numbers in parentheses indicate ranges

our model, we consider this agreement to be worth to note.

5. Conclusions. We propose a scenario of turbulent thermal convection in a rapidly rotating body when the Coriolis force shifts the system to the origin of small scales even at the onset of convection. We also show that the further increase in intensity of heat sources leads to a turbulent regime which is still far from Kolmogorov's case. It appears that the linear analysis predictions at the onset of convection are applicable to our eddy diffusion estimate in the regime of fully developed turbulence. Though the original problem is highly anisotropic, the "isotropic" estimate of eddy diffusion gives a kinetic energy of the system comparable to observations. Note that the introduction of the magnetic field will not change the problem in essence, because at the scales l_E considered the corresponding micro-scale magnetic Reynolds number is $r_m \ll 1$ and the magnetic field decays due to the ohmic dissipation process. On the other hand, it is not yet clear how the west drift velocity relates to the flow at scales l_E, and different interpretations of observations can exist. This question requires the solution of the full dynamo problem.

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