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The Shell Model Approach to the Rapidly Rotating Liquid Bodies

M. Reshetnyak*

Abstract

Applications of the shell model of turbulence to the case of rapidly rotating bodies are considered. Starting from the classical GOY model we introduce the Coriolis force and obtain a $\sim k^{-2}$ spectrum for 3D hydrodynamical turbulence for the free decay regime as well for the regime with external forcing. Additional modifications of the GOY model providing a realistic form of the helicity are proposed.

1 Introduction

The influence of rotation on the properties of the hydrodynamical turbulence is of the great importance. This problem appears in the various geophysical and astrophysical applications and it requires special treatment. So far the direct 3D numerical simulations, where implementation of the Coriolis force is trivial, cannot provide a long enough inertial range of spectrum to reveal its scaling laws, thus different approaches are needed. Though the possibility of transition from 3D-turbulence state to the two-dimensional turbulence due to rotation was already predicted by Batchelor many years ago [1], only now is the qualitative description of this processes developing [2]. Having in mind the similar situation in the MHD turbulence [3], where the magnetic field plays the same role as the rotation in reducing the dimension of the problem, the authors proposed that the Coriolis force introduces a new characteristic time. This time should be used instead of the characteristic turn-over time based on the Kolmogorov's estimate. Then, instead of Kolmogorov's $-5/3$ slope the spectrum will have a -2 slope. It appears that this approach is in agreement with the direct numerical calculations [4] and the experiments [5]. The formal simplicity of this phenomenological approach attracts us to implement it in the more complicated models of turbulence. For this aim we use the well-known homogeneous isotropical GOY shell model (see overview in [6]) and modify it to the case of rotation. The proposed model is tested for the regimes of the free-decay turbulence and for the regime with external forcing. We also consider a situation, where a non-zero average helicity is generated. This regime finds its application in the mean-field dynamo problems.

*Institute of the Physics of the Earth, Russian Acad. Sci., 123995 Moscow, Russia; Research Computing Center of Moscow State University, 119899, Moscow, Russia, e-mail: reshetnyak@ifz.ru

2 Shell model equations

The idea of the shell model approach is to mimic the Navier-Stokes equation by a dynamical system with N complex variables $u_1, u_2, \ldots, u_N$, which represent the typical magnitude of the velocity on a certain length scale. The Fourier space is divided into n_N shells, so that u_n is the velocity difference over length $\sim 1/k_n$, $k_n = 2^n$. Here and after we will refer to the GOY model [6], which has the form:

$$\mathrm{Ro}\,\frac{du_\mathrm{n}}{dt} = \mathrm{Ro}\, ik_\mathrm{n}\Big(u^*_{n+1}u^*_{\mathrm{n}+2} - \frac{\epsilon}{2}u^*_{\mathrm{n}-1}u^*_{\mathrm{n}+1} - \frac{1-\epsilon}{4}u^*_{\mathrm{n}-2}u^*_{\mathrm{n}-1}\Big) - \\ - \mathrm{E}\,k^2_\mathrm{n}u_\mathrm{n} + F_\mathrm{n}, \tag{1}$$

where Ro and E are the Rossby and Ekman numbers, ϵ is a free parameter, * is the complex conjugate, and F is a force. In general, $F = F_c + f$, where F_c is the Coriolis force and f is an external force. In the inviscid limit ($\mathrm{E} \to 0$) and free forcing ($F = 0$) equation (1) conserves the kinetic energy $E_k = \sum\limits_n |u_n|^2$ and $\widehat{H} = \sum\limits_n [\mathrm{sgn}(\epsilon - 1)]^n k_\mathrm{n}^{\alpha(\epsilon)}|u_n|^2$ is an analog of helicity $\widehat{H} = \mathbf{V}\mathrm{rot}\mathbf{V}$ in the 3D case with $\alpha(\epsilon) = -\log_2(|\epsilon - 1|/2)$. In the case of $\epsilon = 1/2$ the dimension of $\widehat{H}$ is equal to the dimension of the hydrodynamical helicity, it is this value of ϵ we consider in the paper.

To introduce effect of rotation in the GOY model we propose that the Coriolis force can be written in the form:

$$F_{\mathrm{c}_\mathrm{n}} = -iC_\mathrm{r}u_n, \tag{2}$$

where C_r is the real constant. It is easy to see, that the work of the Coriolis force (2) is zero ($u^*_n F_{\mathrm{cn}} + u_\mathrm{n} F^*_{\mathrm{cn}} = 0$) and no additional energy is introduced into the system (1–2) at any scale. Having in mind, that in derivation of the shell model equations (1) all external potential forces, as well as pressure, were already excluded using condition of incompressibility ($\nabla \cdot \mathbf{V} = 0$), F_c corresponds to the curl part of the Coriolis force [6].

3 Results of simulation

To analyze behaviour of the hydrodynamical turbulence without forcing we start from the free decay regime ($f = 0$). After some intermediate regime for the case without rotation ($F_\mathrm{c} = 0$) the Kolmogorov's spectrum ($-5/3$) recovers. The inertial spectrum extends up to the Kolmogorov's scale, estimated from the balance of the inertial $\mathrm{Ro}\,k_n u_n^2$ and diffusion $\mathrm{E}\,k_n^2 u_n$ terms: $k_d = \frac{\mathrm{Ro}}{\mathrm{E}} u_n$. As was predicted in [2], introduction of rotation ($F_\mathrm{c} \neq 0$) gives rise to a new time scale $\tau_\mathrm{d} \sim \Omega^{-1}$. For $R_\mathrm{o} \ll 1$, τ_d is already shorter then the characteristic time in the Kolmogorov's regime $\tau_\mathrm{dn} \sim k_\mathrm{n}^{-2/3}$. A simple dimensional analysis leads to the estimate of the rate of the energy dissipation $\varepsilon \sim \tau(k)k^4E^2(k)$, where τ is the characteristic time and $E(k)$ is the spectral energy density. In the case of the Kolmogorov's turbulence $\tau \sim \left(k^3E\right)^{-1/2}$ and $E(k) \sim \varepsilon^{2/3}k^{-5/3}$. If the effect of rotation is sufficient, then substitution of $\tau \sim \Omega^{-1}$ leads to the rotation spectrum law [2]: $E(k) \sim (\Omega\varepsilon)^{1/2}\,k^{-2}$. Starting simulations from the initial field obtained in the non-rotating regime, after a short time period the original Kolmogorov's spectrum splits into two parts with two different slopes. The change in the slope corresponds to $k_\Omega = \frac{C_\Omega^2}{R_\mathrm{o} u_{n_\Omega}}$, where $C_\Omega = 1.22 \div 1.87$ [2]. This estimate can be obtained from balancing the inertial term and the Coriolis force. If for the large scales the Coriolis term is larger then the non-linear term, the spectrum decays as $\sim k^{-2}$. In this case non-linear term does not depend on k and the Coriolis term decays as $\sim k^{-1/2}$. The further behaviour depends on how long the spectrum is and where the Kolmogorov's wave number k_d lies. If $k_\Omega > k_\mathrm{d}$, then the whole spectrum decays as $\sim k^{-2}$.

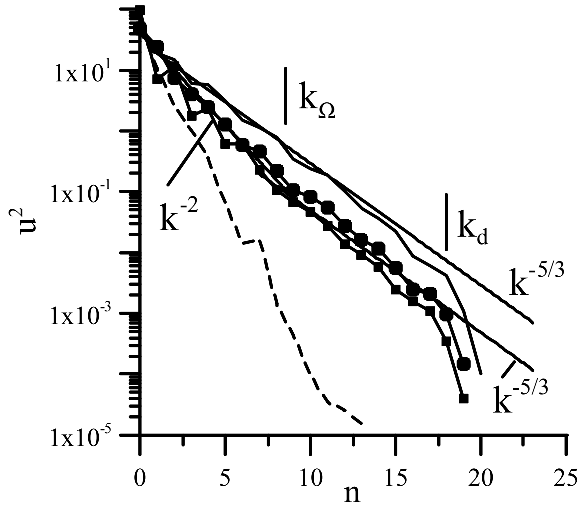

Figure 1: Spectra of the turbulence with external forcing $f_0 = 10^{-2}(1+i)$, $E = 10^{-10}$, $R_o = 10^{-3}$. Solid line corresponds to the regime without rotation, $C_{\rm r} = 0$; line with circles corresponds to the regime with rotation, $C_{\rm r} = 1$ and the Coriolis force defined by (2); the dotted line to $C_{\rm r} = 10$. The line with squares corresponds to the modified Coriolis force (4).

In the other case ($k_\Omega < k_{\rm d}$) the Kolmogorov's spectrum $-5/3$ for $k > k_\Omega$ reappears. As the whole kinetic energy of the system decays in time, the k_Ω moves in to the region of the large wave numbers.

To consider the evolution of the system for time periods longer than the characteristic decay time one needs to provide some source of energy to the system or to introduce an external force f. The results of simulations over the time period $t = 10^3$ with a prescribed force $f = 10^{-2}(1+i)$ at $k_0 = 1$ without rotation are presented in Fig. 1.

Starting from an arbitrary initial velocity field, the system (1) comes to the statistically stable state with Kolmogorov's energy distribution, similar to the free decay case for the moment where the energy is comparable. As was already mentioned above, the direct introduction of rotation setting $C_{\rm r} = 1$ in (2) contradicts to the physics of the problem and to the original Navier-Stokes equation: for the regime of the fast rotation (Ro $\ll 1$), when $V <$ Ro, the balance between the pressure and the potential parts of the forces f and $F_{\rm c}$ holds. Such a balance of the pressure and the Coriolis force, which takes place, e.g., in the Earth's liquid core ($\mathrm{E} = 10^{-15}$, $\mathrm{Ro} = 4 \cdot 10^{-7}$), is called the geostrophic balance (in the present dimensionless units regime the Earth's core situation corresponds to $\mathrm{Ro}\, V < 10^{-3}$). Exclusion of the pressure requires exclusion of the potential parts of all the forces, also. Then, the remaining curl parts of the forces are already of the same order as the inertial term and it is these parts, which are in the r.h.s. of (1). These considerations can be formulated as follows:

$$\begin{cases} C_{\rm r_n} = \mathrm{Ro}\, k_{\rm n}|u_{\rm n}|, & \mathrm{Ro}\, k_{\rm n}|u_{\rm n}|^2 < |u_{\rm n}| \\ C_{\rm r_n} = 1, & \mathrm{Ro}\, k_{\rm n}|u_{\rm n}|^2 > |u_{\rm n}|. \end{cases} \quad (3)$$

In other words, condition (3) means, that for all wave numbers, where the Coriolis force is larger then the non-linear term, it must be compensated by the pressure, and its curl part ($F_{\rm c}$) must be of the same order as the non-linear term. The results of simulations with rotation and $C_{\rm r}$, defined by (3), are presented in Fig. 1. As in the free decay case we observe two regimes with slopes $\sim k^{-2}$ for the small wave numbers and the Kolmogorov's regime $\sim k^{-5/3}$ for the wave numbers $k > k_\Omega$. It is easy to see, that in this model the non-linear term has a white spectrum for $k < k_\Omega$. Due to the condition (3), the curl part of Coriolis term F_c has the same distribution in

the wave space. This state of equipartition was observed after averaging. Analysis of the phases of the non-linear term and the Coriolis term reveals the existence of anticorrelation. This is the reason why the spectrum of the rotating turbulence decreases faster then Kolmogorov's one. In fact, the Coriolis force partially locks the energy transfer from large to small scales. Moreover, the Coriolis force blocks the applied external force, so that $F_{c0} = -f_{c0}$ and suppresses injection of the energy into the system (1) (see demonstration of this suggestion in Fig. 1, regime of turbulence degeneration with $C_r = 10$). The other important point is, that decreasing of the slope of the spectrum caused by rotation is equivalent to increasing of the kinetic energy at the large-scales [4].
We also present development of the model to description of helicity generation in the rotating liquid. Note, that the classical definition of $\widehat{H}$ in the shell model theory does not include direction of rotation of the whole system. To overcome this difficulty we propose, that the Coriolis force should be modified as follows:

$$F_c = -iC_r u(1 + C_1(-1)^n), \tag{4}$$

where C_1 is a real constant. This modification of the Coriolis force does not change the slope of the spectrum (see Fig. 1). From the other point of view, it helps to reproduce regimes with the non-zero mean level of $\widehat{H}$, which are of the great importance in the mean-field dynamo theory [7].
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