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GEOPHYSICS

Rotation of the Solid Core with Account for the Ekman Layer

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The problem of heat convection in the Earth's liquid core and the coupled dynamo problem, which both include the consideration of self-consistent rotation of the solid core, are fundamental geophysical problems. A few currently existing three-dimensional numerical models of the dynamo are based on the mechanisms of thermal and concentration convection [1, 2]. Despite the fact that these models can describe certain large-scale characteristics of the observed values [3, 4], e.g., the spectra of the geomagnetic field and the direction of the solid core rotation, the absolute values of the parameters used in these models are *a fortiori* significantly different from the Earth's actual parameters. One of the main problems that we face during the development of models of convection in the Earth's liquid core, is the existence of thin Ekman layers at the contact of the liquid core with the solid core and mantle, whose thicknesses are of the order of $\delta = E^{1/2}$, where $E \sim 10^{-16}$ is the Ekman number. In other words, the resolution of these layers would require 10^8 grid points, at least in one radial (normal to the surface) direction. It is clear that this problem is beyond the capabilities of present-day computers and will not likely be solved using direct methods in the near future. We note that it became possible to obtain the regimes with $E = 2 \cdot 10^{-6}$ only by applying a nonuniform grid in model [1]. In order to overcome this difficulty, the authors of [5, 6] neglected the fluid viscosity and inertial terms in the main volume during the solution of a two-dimensional axial-symmetric problem of the $\alpha\omega$-dynamo. These simplifications, which are applicable in the theory of the dynamo, allowed them to simplify the problem and reduce it to a system of algebraic equations. In order to ensure that the velocity field obtained by the numerical integration of the system of equations would satisfy the zero boundary conditions, they used information about the analytical solution in the Ekman layer where viscous forces are already compatible with the Coriolis and magnetic forces. This approach allowed them to gain a significantly better description of the solution near the solid boundaries.

Together with the advantage of the inviscid approximation mentioned above, sufficiently strict restrictions exist for its application. As was mentioned above, inertial terms are also neglected, in addition to the viscous ones, within the framework of this approach. This can be done on the basis of comparing the characteristic times of convection and changes in the magnetic field (the latter is a few orders of magnitude greater than the former). Since, according to direct and indirect observations, the energy of the magnetic field between inversions exceeds the energy of kinetic motions by three orders of magnitude (see, for example, [3]), and since Lorentz forces dominate in the system, the hydrodynamic properties "instantaneously" adjust to the magnetic field. We reiterate that the ratio of the characteristic convection time to the magnetic time matches the Rossby number $\mathrm{R}_0 \sim 4 \cdot 10^{-7}$, and the inviscid model is self-consistent. However, during the attenuation of the magnetic field (including the local one) and the appearance of regions where the magnetic field is not a field of force, the initial approximations may not be justified. The situation is aggravated by the fact that, according to models [7, 8], the appearance of $\sim E^{-1/3}$ vertical cylindrical structures (Busse columns) is predicted in the absence of the magnetic field, in which viscosity and inertial terms can play a significant role.

In this work, we propose a pioneer method (based on the initial system of equations for thermal convection and the Boussinesq approximation) that allows us to calculate the moment of viscous forces acting on the solid core with account for the Ekman solutions in the layers. The estimates of the rotation velocity of the inner core with respect to the mantle were obtained in the course of studies at different regimes of convection. It is shown that the account for the Ekman layers can lead to significant changes in the hydrodynamics of the liquid core. This approach can be easily extended to the case of a compressible fluid and an Ekman–Hartman layer in the presence of the magnetic field.

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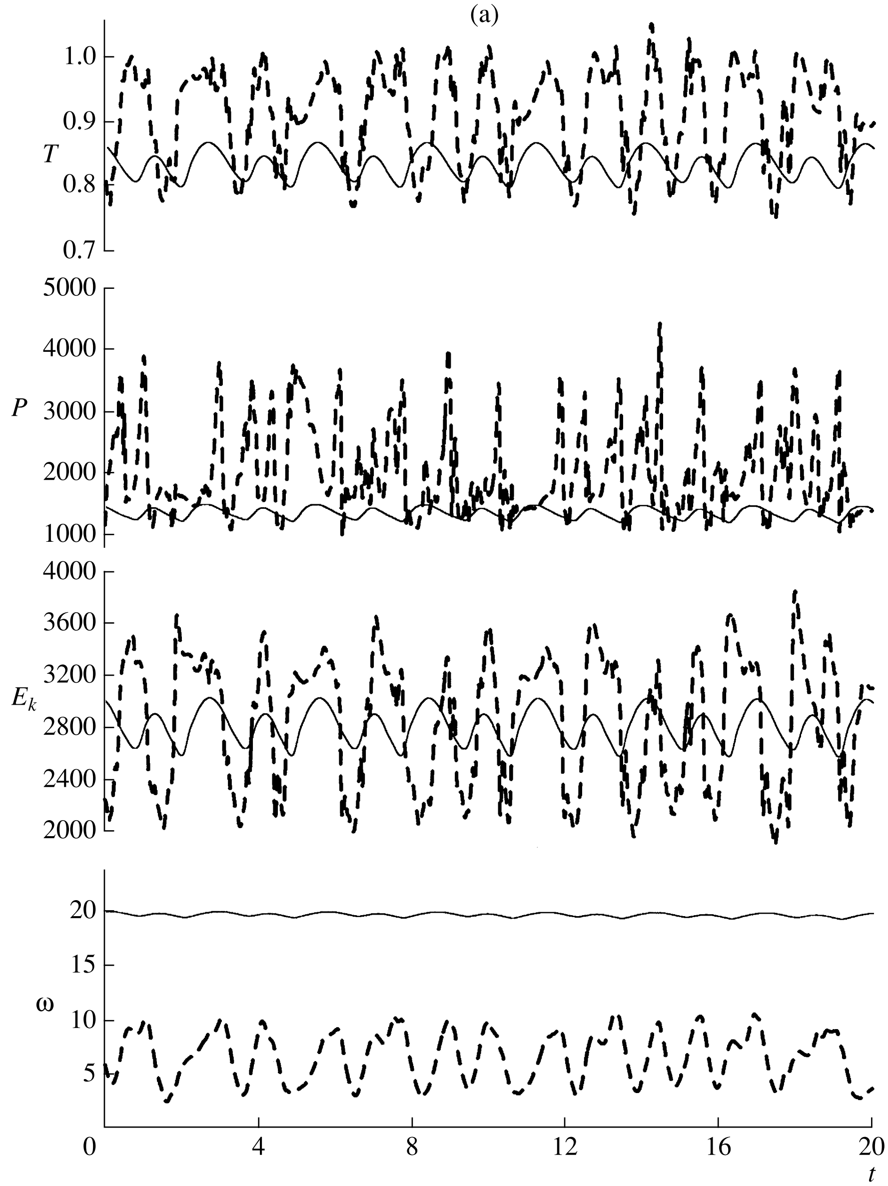

Fig. 1. Evolution of the volume mean values: T, P, kinetic energy E_k, and angular velocity ω. All values, including time, are given in dimensionless units; $R_0 = 1$, $Ra = 10^4$, $E = 10^{-3}$ (a), $E = 10^{-5}$ (b). Dashed line shows the evolution without account for the Ekman layer; solid line shows the evolution with account for the Ekman layer.

Thermal convection in the Earth's liquid core ($r_i < r < r_0$) with radius $L = r_0$ can be described by the Boussinesq equation system for an incompressible fluid ($\nabla \cdot \mathbf{V} = 0$) in the dimensionless form including the equation of motion

$$R_0\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right) = -\nabla p + \mathbf{F} + E\nabla^2\mathbf{V} \qquad (1)$$

and heat transfer for the deviations of the temperature field T from the given profile T_0

$$\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla)(T + T_0) = q\nabla^2 T, \qquad (2)$$

where $\mathbf{F}$ is the sum of the Coriolis and buoyancy forces $\mathbf{F} = -\mathbf{1}_z \times \mathbf{v} + q\mathrm{Ra}\,Tr\mathbf{1}_r$. The following notations are used in Eqs. (1) and (2): $E = \dfrac{\nu}{2\Omega L^2}$ is the Ekman number, $q = \dfrac{\kappa}{\eta} = 1$, ν is the coefficient of kinematic viscosity of the fluid, Ω is the angular velocity of the Earth's diurnal rotation, $L = r_0$ is the characteristic scale, $T_0 = \dfrac{r_i/r - 1}{1 - r_i}$ is the given adiabatic temperature profile, $\mathrm{Ra} = \dfrac{\alpha g_0 \Theta L}{2\Omega\eta}$ is

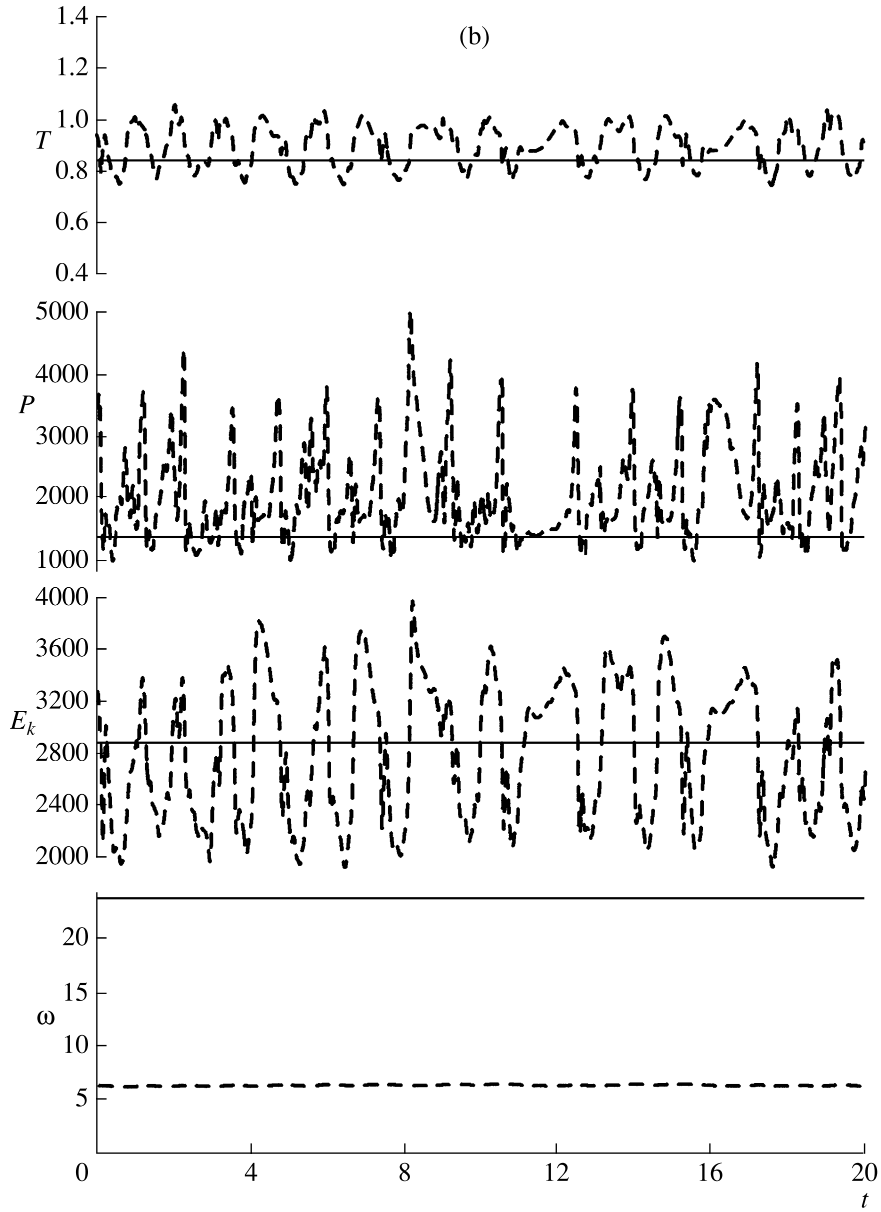

Fig. 1. (Contd.).

the Rayleigh number, α is the coefficient of volumetric expansion, g_0 is the free-fall acceleration, κ is the coefficient of thermal conductivity, η is the coefficient of magnetic diffusion, and Θ is the unit of temperature measurement. The units $\frac{L^2}{\eta}$ and $\mathbf{V} - \frac{\eta}{L}$ were used for measuring time and velocity, respectively, (*r*, θ, φ) is the spherical coordinate system, and *z* is the unit vector in the direction of the Earth's rotation. The application of this normalization allows us to extend the problem to the magnetic case by including the equation for induction.

In the general case, the inner core can freely rotate around the vertical axis under the influence of viscous forces. The equation for the moment of impulse written in the form of the integral over the solid core surface (*S*) with respect to the tensor of viscous stresses τ takes the following form:

$$\mathrm{R}_0 I\frac{\partial\omega}{\partial t} = r_i\oint_S \tau_{r\varphi}\big|_{r = r_i}\sin\theta ds, \qquad (3)$$

where ω is the sought angular velocity of the solid core rotation and $I = \frac{8}{15}\pi r_i^5$ s the moment of inertia of the solid core.

Equation system (1)–(4) is closed by zero boundary conditions for temperature (*T*) fluctuations and by the zero transport and nonslip conditions at solid boundaries for velocity field **V**. It is easy to see from Eq. (1) that when E tends to zero the solution has boundary (Ekman) layers at $r = r_i$, r_0 and the problem becomes nontrivial for calculations. Despite the fact that the solution can be easily found in the analytical form when the velocity field is specified at the boundary with the main volume, the problem of correlation of solutions in the layers and the main volume is not solved yet and requires the joint solution of both problems.

The idea behind the approach suggested is to use the information about the behavior of the velocity field in the Ekman layer to calculate the tensor of viscous stresses on the solid core surface.

Let us present the solution in a layer as $\mathbf{V} = \mathbf{V}^0 + \hat{\mathbf{V}}$, where $\mathbf{V}^0$ is the solution in the main volume (constant within the layer) and $\hat{\mathbf{V}}$ is the sought solution at which the relative velocity of the fluid at the solid boundary turns to zero, and $\hat{\mathbf{V}} = 0$ far from the wall.

According to the theory of the Ekman layer (see, for example, [9, 10]), the solution is sought from the balance condition between the Coriolis and viscous forces. If we neglect the tangential derivatives compared to normal ones, Eq. (1) for the tangential components of velocity can be written after simple transformations in the following form

$$-\hat{V}_\varphi \cos\theta + E\frac{\partial^2 \hat{V}_\theta}{\partial r^2} = 0,$$
$$\hat{V}_\theta \cos\theta + E\frac{\partial^2 \hat{V}_\varphi}{\partial r^2} = 0, \tag{4}$$

whose solution is

$$\hat{V}_\theta = -((V_\theta^0 - V_\theta^s)\cos(\alpha x) - (V_\varphi^0 - V_\varphi^s)\sin(\alpha x))e^{-\alpha x},$$
$$\hat{V}_\varphi = -((V_\varphi^0 - V_\varphi^s)\cos(\alpha x) + (V_\theta^0 - V_\theta^s)\sin(\alpha x))e^{-\alpha x}, \tag{5}$$

where $\alpha = \sqrt{\frac{|\cos\theta|}{2E}}$ is the value inverse to the dimensionless thickness of the boundary layer, x is the distance from the solid boundary, and V^s is the solid boundary velocity ($V_\theta^s = 0$). The relations presented here describe the behavior of the tangential components of velocity in a layer and can be used for formulating boundary conditions for the problem in the main volume of the fluid core and also to determine the component of the tensor of viscous stresses $\tau_{r\varphi}$ in (3).

The SIMPLE method described in detail in [11] was used to solve Eqs. (1)–(3). The upwind scheme with a grid ($15 \times 15 \times 15$) was used to approximate nonlinear convective terms in the equations. One of the problems that arises in contriving the finite-difference schemes in a spherical coordinate system is the formulation of additional boundary conditions at the axis and in the center (see [12]). This problem is easily bypassed within the SIMPLE method, where the fluxes are considered across control-volume surfaces. Since the fluxes across elementary squares at the axis and in the center are equal to zero, it is not required to specify the fields in these regions.

The evolution of certain characteristics of the solution of equation system (1)–(3) for different values of parameters is shown in Fig. 1a. Traditional right-side three-point schemes of the second order of accuracy were used to calculate the derivatives in $\tau_{r\varphi} = E\left(\frac{\partial V_\varphi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{V_\varphi}{r}\right)$ (see [13] for details).

The evolution of the same characteristics using the solutions in the layers is presented in Fig. 1b for comparison. In this case, $\tau_{r\varphi}$ was calculated from relations (5). The advantage of this method is in the fact that the value of velocity at only one point along r beyond the Ekman layer is needed for calculating the tensor of stresses at the surface of the solid core. However, in the case of the traditional finite-difference approximation of the layer solution, we should take care that the number of points is sufficient for describing the solution with the required accuracy, which already becomes difficult at small E.

Comparative analysis of the figures indicates that the account for asymptotics in a thin layer influences the hydrodynamic properties of the problem as a whole, while the difference in the behavior of the integral characteristics of the problem increases with the decrease of the Ekman number. Almost in all cases, the dispersion of the characteristics decreases if Ekman asymptotics are used. The solution becomes smoother and even constant in time. The latter reflects the fact that the usual methods of the applied grid are not enough to approximate the viscous forces in a layer and it is necessary to apply more detailed information about the behavior of the solution in the layer.

Additional calculation for the stationary case ($E = 10^{-5}$) with a specified angular velocity of the solid core rotation ω was carried out during the analysis of the results. As expected, the solution remained stationary.

We note that there is no problem in determining the surface force in the calculation of the moment of magnetic force, since only the field itself and not its spatial derivatives is included in the expression for the moment [10] (see also [14]).

In conclusion, we note that this study does not claim to solve the full self-consistent problem, where layered solutions would also be taken into account in Eq. (1). It only demonstrates the importance of accounting for the Ekman layers in the calculation of the viscous forces applied to the solid core. On the other hand, the effect related to the solid core rotation is integral and can significantly influence the hydrodynamic properties of the core as a whole. Numerical experiments carried out by the author showed that additional account for the solution in the Ekman layers using the method of near-wall functions (see, for example, [15]) in Eq. (1) did not have any significant effect on the behavior of the solution at $E \to 0$.

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