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Correlation characteristics of the secular variation field

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Abstract. Different methods of constructing a spatial autocorrelation function of the secular variation field on the Earth's liquid core surface are proposed. The Earth's liquid core is found to feature magnetic field irregularities with a characteristic scale of not over $t_W \simeq 1500$ km. This scale is shown to be likely to correspond to the length of the magnetic loops in the model of *Ruzmaykin et al.* [1989].

Introduction

Numerous experimental data and theoretical calculation results on the structure and origin of the secular geomagnetic variations have been accumulated to date. The basic idea, suggested first by *Braginskiy* [1972], is that their source is found at the mantle-core boundary. Convective motions of conductive plasma in this layer tend to generate variations in the magnetic field at the Earth's surface. The study of the morphology of the variations and their sources can conditionally be divided into two approaches.

According to one of these methods the magnetic field is extrapolated to the Earth's core, and based on certain assumptions of hydrodynamic properties of a conductive medium (e.g., geostrophy of the flow), maps of flows at the mantle-core boundary are plotted. This approach entails the well-known difficulties of solving ill-posed problems and also of choosing a magnetohydrodynamic model of the boundary layer at the mantle-core boundary [*Gubbins*, 1991].

The second approach is a statistical one. Within its scope, spatial and temporal spectral characteristics of the magnetic field and its variations are studied. To expand the observed field in terms of spherical functions at a fixed moment of time, energy distribution of the field itself and its variation over both the Earth's surface and the liquid core surface has been thoroughly investigated [*Golovkov and Chernova*, 1988; *Lowes*, 1974]. In particular, for the secular variation field this distribution over the core surface is substantially a white noise. The nondecay of spectrum and increase in errors in the region of large wave numbers makes it difficult to analyze the results [*Rotanova*, 1989]. Another method of studying the magnetic field statistical properties is to investigate its correlation characteristics. We will use this method in the present study.

One of the models explaining the random behavior of the secular variation field is the fluctuation dynamo model [*Ruzmaykin et al.,* 1989], which holds that a random magnetic field can be generated in the random flow of conductive medium with $\mathrm{Rm} \geq 10^2$. Within this model the simplest characteristics of the random field are its correlation functions. *Kliorin et al.* [1988], using asymptotic methods, constructed correlation functions of a random magnetic field in a uniform, isotropic, turbulent medium that were determined by a single parameter, the magnetic Reynolds number Rm. *Pilipenko and Sokolov* [1991a, b] and *Sokoloff and Zinchenko* [1992] studied the tensor of a random magnetic field and, using asymptotic methods, investigated its propagation through a two-dimensional layer of vacuum, and *Reshetnyak et al.* [1993] constructed correlation functions of the secular variation field from observational data [*Langel et al.,* 1982]. The obtained qualitative agreement with the theoretical model is far from adequate because of the instability of the inverse problem to the low disturbances in the input data, and therefore a numerical solution of the problem is required. The possibility of solving the inverse problem not for the magnetic field itself but rather for its correlation functions with already smoothed random errors also leads to enhanced accuracy.

In this study, we attempted to solve the inverse problem through the definition of the autocorrelation function for the normal component $\dot{H}_{zz}$ of the secular variation field $\dot{\mathbf{H}}$ on the Earth's liquid core surface, using the data of *Langel et al.* [1982] and *Reshetnyak et al.* [1993] (the dot mark means the time derivative). The obtained result is compared with an analogous result obtained by extrapolation of the field itself up to the core, and the focuses of the secular variation on the core surface are shown to have a scale that is substantially larger than the field tube cross-section dimension, with which *Ruz-*

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maykin et al. [1989] intended to identify the focuses.

Autocorrelation Function of a Random Magnetic Field

Let the secular variation field $\dot{\mathbf{H}}$ induced by convective flows at the core-mantle boundary be random [*Ruzmaykin et al.,* 1989]. For the sake of simplicity we approximate the core-mantle interface by a surface $\sigma(z=0)$. Because of the boundary conditions for $\dot{\mathbf{H}}$, only the normal field component H_z^0 is not zero on the core surface σ. Assume further that the distribution of the component H_z^0 is uniform. It can then be described using a correlation function that depends only on the modulus of distance between points on the σ surface:

$$H_{zz}^0(\rho) = \langle H_z^0 H_z^0 \rangle \quad (1)$$

where the angular brackets denote the statistical averaging and ρ is the distance between points on σ. It is assumed that variations are not related to the mean field, so $\langle H_z^0 \rangle = 0$. As shown by *Reshetnyak et al.* [1993], the autocorrelation function H_{zz} at a distance L from the core σ is described by the expression

$$H_{zz}(a,L) = \frac{L}{\pi}\int\limits_{\sigma} \frac{H_{zz}^0(\rho)\rho d\rho d\varphi}{(R_{a\rho}^2+4L^2)^{3/2}} \quad (2)$$

where $R_{a\rho}^2 = \rho^2 + a^2 - 2\rho a \cos\varphi$ and also meets the condition following from the nondivergence of the magnetic field:

$$\int\limits_0^\infty H_{zz}^0(\rho)\rho d\rho = 0 \quad (3)$$

For the sake of convenience we will hereinafter denote the distance between points on the core surface by ρ and the distance on the Earth's surface by a.

Kliorin et al. [1988] suggested a model of the autocorrelation function H_{zz}^0, a random magnetic field in a uniform, isotropic, conductive medium. In accordance with the asymptotes given, the minimum value of $H_{zz}^0 \cong -\mathrm{Rm}^{-5/4}$, and the distance ρ_1 at which the function $H_{zz}^0(\rho_1) = 0.8H_{zz}^0(0)$ is of the order of $\mathrm{Rm}^{-1/2}$; in this context, $\mathrm{Rm} = lv/\nu$, where $l = 1700$ km is the size of a convection cell, $v = 0.1$ cm s^{-1} is the velocity of density convection, $\nu = 2\times10^4$ cm^2 s^{-1} is the magnetic diffusion coefficient. At $\mathrm{Rm} \cong 10^3$ the minimum will amount to $\cong 2\times10^{-4}$ of the amplitude of the function H_{zz}^0. Reconstruction of the correlation function of a random magnetic field on the core surface is expediently carried out in terms of the function W_Π^0 determined by the relationship

$$W_\Pi^0(\rho) = \frac{6}{\rho^2}\int\limits_0^\rho H_{zz}^0(\zeta)\zeta d\zeta \quad (4)$$

According to (3), W_Π^0 is a monotonically decreasing and always positive function. In view of (4), (2) takes the form

$$H_{zz}(a,L) = \frac{L}{2\pi}\int\limits_{\sigma} \frac{W_\Pi^0(\rho)\rho^2(\rho - a\cos\varphi)\rho d\rho d\varphi}{(R_{a\rho}^2+4L^2)^{5/2}} \quad (5)$$

Figure 1a shows a family of functions W_Π^0 for the range of values of $\mathrm{Rm} = 10^2 - 10^4$. The corresponding family of curves H_{zz} on the Earth's surface is shown in Figure 2. Figure 3 shows the magnetic Reynolds number Rm dependence of the correlation function H_{zz} which can be described by the formula $H_{zz}(0) = 2\times10^{-3}\mathrm{Rm}^{-0.8}$ at $H_{zz}^0(0) = 1$. (Defined on the Earth's surface was a model, normalized to its maximum value, autocorrelation function $H_{zz}^0(0)$ [*Ruzmaykin et al.,* 1989] as a function of Rm).

Hereinafter, we will solve the inverse problem: from the prescribed function H_{zz} on the Earth's surface, calculated by experimental data [*Pilipenko and Sokolov,* 1991b]; we will find the function $W(0,\Pi)$ on the Earth's liquid core surface.

Observational Data

Reshetnyak et al. [1993] suggested an algorithm for constructing correlation functions of the secular variation field on the Earth's surface and, in particular, the autocorrelation function $\dot{H}_{zz}$, equal by definition to $\langle \dot{H}_z \dot{H}_z \rangle$, from the observational data of *Langel et al.* [1982]. The function was constructed on the assumption of a spheric symmetry of the correlation functions $\dot{H}_{ij}$, so that these depend only on the distance between points a. The algorithm is essentially as follows: One randomly takes a couple of points on the sphere, such as to uniformly cover the sphere all over its area. Then, using a spherical harmonic analysis, one computes the components of the secular variation field $\dot{\mathbf{H}}$ at these points and constructs the correlation functions $\dot{H}_{ij}(a)$ therewith. Unlike *Tikhonov and Arsenin* [1979], in computing the functions $\dot{H}_{ij}$ at point a, the number of points is defined from the condition that the median distance between these be not less than the distance a. This correction leads to a more realistic estimation of errors in $\dot{H}_{ij}$. The component $\dot{H}_{zz}$, normalized to the maximum value 4.8×10^3 (nT/year)^2, and its standard deviation are shown in Figure 2.

Note that relationship (3) is obtained, first, for the magnetic field itself and not for its time derivative and, second, on the assumption of a two-dimensional nonconductive mantle. As for the former condition, it is expanded to the case of secular variation by virtue of nondivergence of the field $\dot{\mathbf{H}}$. Let us consider in more detail the twodimensional mantle condition. Substituting into (2) $\dot{H}_{zz}$ and $\dot{H}_{zz}^0$ for H_{zz} and H_{zz}^0, respectively,

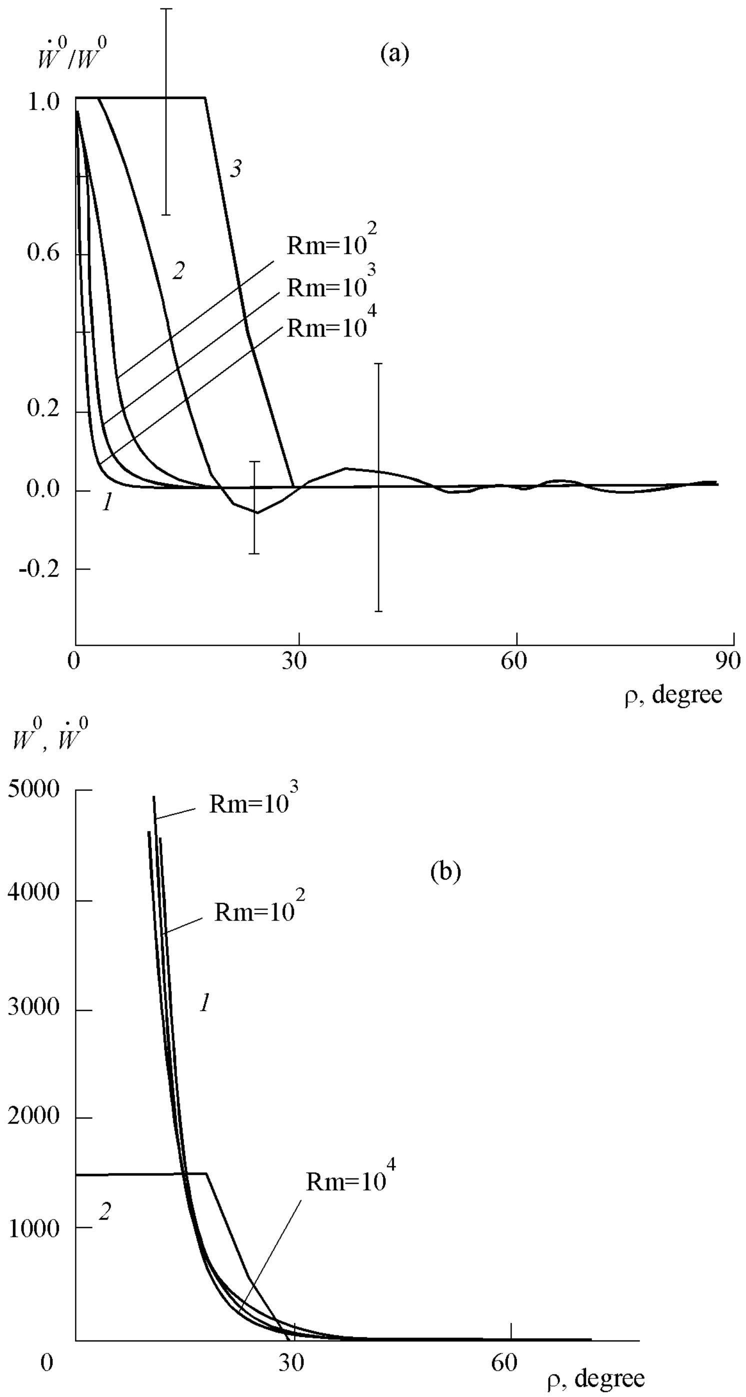

Figure 1. (a) Autocorrelation functions normalized to their maximum value and standard deviation for curves 2 and 3 on the Earth's liquid core surface. Curve 1 indicates a family of model functions W^0 depending on the magnetic Reynolds numbers $\mathrm{Rm} = 10^2, 10^3, 10^4$; curve 2 indicates W^{rest} calculated by analytically expanding the secular variation field to the core surface; and curve 3 indicates W^0 obtained by solving the integral equation (9) followed by transformation (10). (b) Nonnormalized autocorrelation functions on the liquid core surface at a defined unit correlation function on the Earth's surface is shown. Curve 1 indicates a family of model functions W^0 depending on the magnetic Reynolds number $\mathrm{Rm} = 10^2, 10^3, 10^4$; curve 2 indicates W^0 obtained by solving the integral equation (9) followed by transformation (10).

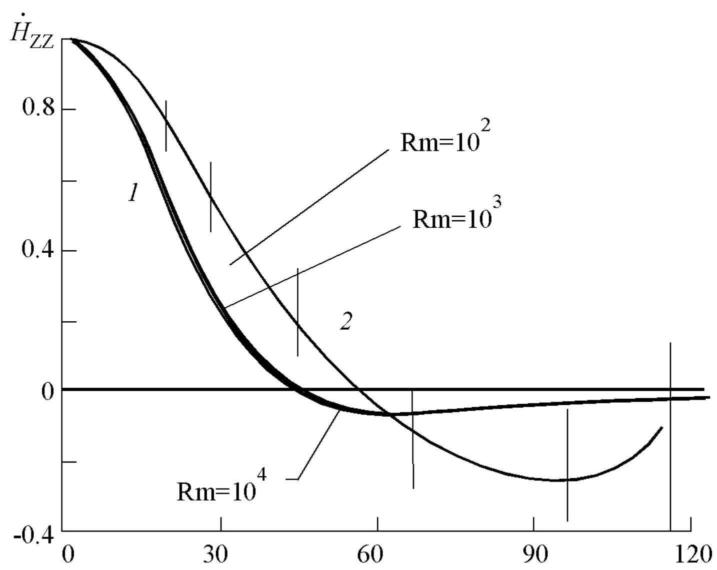

Figure 2. Autocorrelation functions and the standard deviation for curve 2 (Figure 1) on the Earth's surface. Curve 1 indicates a family of curves $\dot{H}_{zz}$ at a family of $\dot{W}^0$ defined on the core surface (Figure 1a and curve 1) for the magnetic Reynolds numbers Rm = 10^2, 10^3, 10^4; curve 2 indicates H_{zz} calculated from experimental data [*Pilipenko and Sokolov,* 1991b; *Sokoloff and Zinchenko*, 1992].

and integrating over the variable a yields

$$\int_0^\infty H^0_{zz}(a)ada = \int_0^\infty H^0_{zz}(\rho)\rho d\rho \qquad (6)$$

Thus condition (3) is invariant relative to parallel transport and is valid on any plane σ^1 parallel to the plane σ of the core-mantle interface and found at a distance L therefrom. Numerically solving (6) by using observational $\dot{H}_{zz}$ data yields

$$\delta_1 = \int_0^\infty \dot{H}_{zz}(a)\, a\, da / \int_0^\infty \dot{H}^0_{zz}(a)\, a\, da = -0.5$$

where the point R is determined from the condition $\dot{H}_{zz}(R) = 0$. The error δ_1 characterizes the error in the two-dimensional model (2). Condition (6) and δ_1 make it possible to significantly confine the class of functions $\theta \ni H_{zz}$, wherein the solution is sought. For a sphere an analogue to (3) is

$$\int_0^\pi \dot{H}^0_{zz}(\varphi) \sin\varphi d\varphi = 0 \qquad (7)$$

Unlike (3), where the function $\dot{H}^0_{zz}$ was considered in an unlimited, uniform, isotropic space, relationship (7) takes into account the transition to a real, spherical core of the Earth. In this context the element of area $\rho d\rho$ in (3) changes to $\sin\varphi d\varphi$.

A numerical estimation yields

$$\delta_2 = \int_0^\pi \dot{H}^0_{zz}(\varphi) \sin\varphi d\varphi / \int_0^\beta \dot{H}^0_{zz}(\varphi) \sin\varphi d\varphi = -0.05$$

which is in magnitude far less than δ_1, where point β is determined from the condition $\dot{H}_{zz}(\beta) = 0$.

Since the condition of positive W^0_Π is satisfied only at $\delta_1 \geq 0$ we will introduce a new function W^0_S, as in (4)

$$W^0_S(\rho) = \frac{6R_c}{\rho^2} \int_0^\rho H^0_{zz}(\zeta) \sin\left(\frac{\zeta}{R_c}\right) d\zeta \qquad (8)$$

where integration is carried out up until $\rho_{\max} = \pi R_c$ and R_c is the Earth's liquid core radius. The function W^0_S, thus introduced, is positive and monotonically decreasing. Substituting (8) into (2) yields an equation for W^0_S, similar to (5):

$$\dot{H}_{zz}(a, L) = -\frac{1}{6R_c} \int_S \frac{\dot{W}^0_S(\rho)\rho^2}{\sin(\rho/R_c)} K(\rho, a, L)\rho d\rho d\varphi \qquad (9)$$

where

$$K(\rho, a, L) = \frac{1}{R^{3/2}_{a\rho}} \times$$

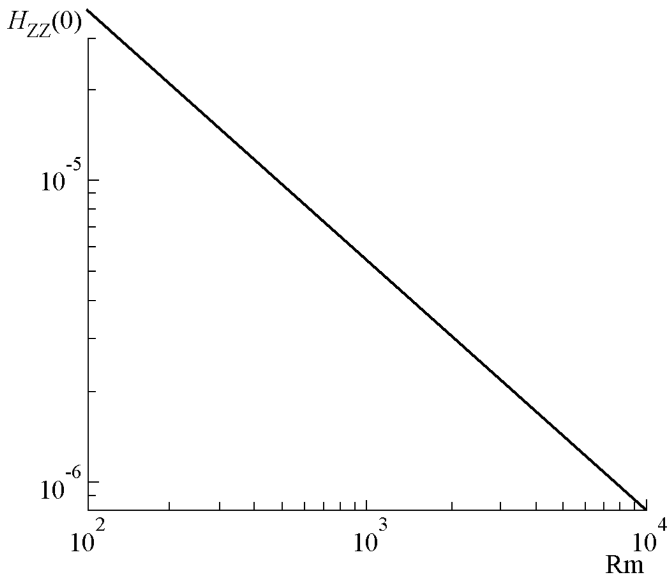

Figure 3. Dependence of the amplitude of the autocorrelation function $H_{zz}(0)$ on the Earth's surface on the magnetic Reynolds number Rm.

$$\times\left(1-\frac{\rho}{R_c}\cot\left(\frac{\rho}{R_c}\right)-\frac{3\rho(\rho-a\cos\varphi)}{R_{a\rho}}\right)$$

The change from $\dot{W}_S^0$ to $\dot{W}_\Pi^0$ can always be performed, using (4) and (8), by formula

$$\dot{W}_\Pi^0(\rho)=\frac{\dot{W}_S\rho}{R_c\sin(\rho/R)}- \qquad (10)$$

$$-\frac{1}{R_c\rho^2}\int\limits_0^\rho\frac{\dot{W}_S^0(\eta)\rho^2}{\sin(\eta/R_c)}\left(1-\frac{\eta}{R_c}\cot\left(\frac{\eta}{R_c}\right)\right)d\eta$$

The limiting change at $\rho_{\max}\to 0$ yields $\dot{W}_\Pi^0\to\dot{W}_S^0$.

Inverse Problem

Thus the problem of defining a spatial autocorrelation function $\dot{W}_\Pi^0$ of the secular variation field on the core surface has been reduced to solving (9) for $\dot{W}_S^0$ with subsequent transformation (10). Equation (9) is a first-order Fredholm equation and belongs to the class of ill-posed problems, where a small disturbance of input data on $\dot{H}_{zz}$ leads to a great change in solving $\dot{W}_S^0$. In the operator form, (9) can be represented as $\dot{H}_{zz}=\hat{A}\dot{W}_S^0$, where $\hat{A}$ is a linear, quite continuous operator in a certain infinite-dimensional functional space. As is shown, for instance, by *Tikhonov and Arsenin* [1979], the operator $\hat{A}^{-1}$, which is reverse to the operator $\hat{A}$, does not show continuity for the Fredholm equation of the first order, whereby (9) requires regularization for its solution [*Tikhonov and Arsenin*, 1979].

Stable approximations for solving ill-posed problems are based on the use of additional information about the solution sought. In our case, it is known a priori that the rigorous solution belongs to a compact set. In this case, the inverse operator $\hat{A}^{-1}$ turns out to be continuous, and a uniform convergence of a sequence of approximations to the rigorous solution of the problem is guaranteed. *Tikhonov et al.* [1990] have shown that the information on the rigorous solution being monotonic and limited is enough for the solution to be stated as belonging to a compactum. The function $\dot{W}_S^0$ meets all these requirements.

To solve (9), we used the method of conjugate gradients with mapping to a nonnegative set. Figure 1a shows the obtained solution of $\dot{W}_\Pi^0$ and its standard deviation. Numerical estimates show that within the accuracy of initial data, transformation (10) introduces no significant changes in going from $\dot{W}_S^0$ to $\dot{W}_\Pi^0$. The amplitude of the function $\dot{W}_\Pi^0$ exceeds 1.8×10^3 times the amplitude of the function $\dot{H}_{zz}$ on the Earth's surface. Since, according to (8), at

$$\rho\to 0,\quad \dot{H}_{zz}^0(\rho)\cong\frac{1}{3}\dot{W}_\Pi^0(\rho),\quad \dot{H}_{zz}^0(0)\cong 630\dot{H}_{zz}(0)$$

which is equivalent to a 25-fold enhancement of the rms magnitude of the field on the surface of the core. Let as introduce an integral scale $\dot{W}^0$:

$$L_W=1/\dot{W}^0(0)\int\limits_0^\infty\dot{W}^0(\rho)d\rho$$

A numerical estimate gives $L_W \cong 1500$ km. The 1500-km distance on the core surface corresponds to 25°. Let us estimate the relation of the magnetic energy of the field variation

$$M = \frac{\dot{H}_{zz}(0)}{8\pi}\tau^2$$

(where $\tau \simeq 100$ years is the characteristic time of the field variation) to the kinetic energy,

$$K = \rho v^2/2 : MK \cong 2$$

which corresponds approximately to the equidistribution hypothesis.

For comparison, we also consider the correlation function $\dot{W}^0$ on the core surface (curve 2 in Figure 1a), constructed by analytically expanding the magnetic field to the core surface using the spherical harmonic analysis data ($n = 14$) of the method described in the preceding paragraph. The obtained solution is evidence for a 120-fold enhancement of the secular variation field and the presence of a correlation scale $aL_W = 600$ km. Note that the function $\dot{W}^0$ is obtained as an alternating function, which contradicts its physical sense. This entails larger errors in extrapolating the time-derivative Gaussian coefficients $\dot{g}_n^m$, $\dot{h}_n^m$, and nonstationarity of the secular variation field with time. Since a 600-km-scale resolution requires a Gaussian series with $n \geq 18$, the obtained solution is rather crude. Note that in the first of the methods proposed above, the high-frequency information on the Earth's surface (essentially the most error-prone) was filtered in constructing the correlation function $\dot{H}_{zz}$. Therefore the solution on the Earth's surface $\dot{W}^0$, too, has a lower amplitude and a larger correlation scale than the $\dot{W}^0$ obtained from the spherical harmonic analysis.

Let us compare the obtained correlation function $\dot{W}^0$ with the model results of *Kliorin et al.* [1988]. The horizontal plateau of curve 3 in Figure 1a may correspond to an unresolved (considering the accuracy of input data) correlation scale. A scale of the order of 100 km may exist and still make no appreciable contribution to the observed field on the Earth's surface. Note that the normalization adopted in Figure 1a fails to adequately characterize the properties of the correlation functions W^0 and $\dot{W}^0$. Therefore we suggest another normalization, namely, at a defined unit correlation function H_{zz} and $\dot{H}_{zz}$, respectively, on the Earth's surface (see Figure 1b). Now a portion $\rho \in (15 - 25°)$ is seen to exist, where the functions differ only slightly from one another. The reconstructed correlation function $\dot{W}^0$ is close to theoretical values, beginning with distances greater than L_W (Figure 1b). Thus the obtained value of L_W can be considered an upper bound on the correlation scale on the Earth's core surface. Note that *Ruzmaykin et al.* [1989] hold this scale to correspond to the magnetic loop length $l \simeq 1700$ km. The 100-km scale on the core surface corresponds to an angular distance of 1.5°. Resolution of such scales requires using Gaussian coefficients $\dot{g}_n^m$, $\dot{h}_n^m$ with numbers n greater than $180° : 1.5° \simeq 120$, which means that the magnetic field magnitude should be far greater than in characteristic fields with scales of 10^3 km. Since the magnitude of small-scale fields is substantially restricted by the equidistribution hypothesis, the methods proposed above fail to discern 100-km scales.

Conclusions

The obtained results are evidence for irregularities of the magnetic field in the Earth's liquid core, with a characteristic scale not over $L_W \simeq 1500$ km, which does not contradict the existence of structures having smaller scales but featuring an amplitude not high enough to be measured on the Earth's surface.

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