A GRID-SPECTRAL METHOD OF THE SOLUTION OF THE 3D KINEMATIC GEODYNAMO WITH THE INNER CORE

P. HEJDA

Geophysical Institute, Acad. Sci. Czech Rep., Prague, Czech Republic[1]

M. RESHETNYAK

Institute of the Physics of the Earth, Russian Acad. Sci, Moscow, Russia[2]

Summary: *A 3D kinematic geodynamo model in a sphere with the conductive solid inner core is considered. The 3D magnetic field and velocity field are resolved in the physical space for r- and θ-coordinates, whereas the sin- and cos-decomposition is applied to the φ-coordinate. The additional boundary conditions for the case of non-zero velocity field on the boundaries of the liquid spherical shell and for different magnetic diffusivities of the inner and outer core are applied. The computer code was tested by free decay mode solutions and comparisons were made also with results reported by other authors. This work is a part of a project to study 3D inviscid geodynamo models.*

Key Words: hydromagnetic dynamo, induction equation, grid methods

1. INTRODUCTION

Due to the very complexity of 3D geodynamo equations, the traditional approach to geodynamo problem is numerical. From the pioneering calculations of the kinematic dynamo by *Bullard and Gellman (1954)* to the most advanced 3D hydrodynamic simulations by *Glatzmaier and Roberts (1995)* or *Kuang and Bloxham (1997)* the numerical process was based on the decomposition of the magnetic and velocity fields into toroidal and poloidal parts $\mathbf{B} = \mathbf{P} + \mathbf{T} = \nabla \times (\mathcal{T}\mathbf{e}_r) + \nabla \times \nabla \times (\mathcal{P}\mathbf{e}_r)$ and similarly for $\mathbf{V}$. In the 2D models this decomposition takes a simplier form $\mathbf{B} = \mathbf{P} + \mathbf{T} = \nabla \times (\mathcal{T}\mathbf{e}_\varphi) + \nabla \times (\mathcal{P}\mathbf{e}_\varphi)$. The scalars $\mathcal{T}$ and $\mathcal{P}$ were usually expanded into spherical harmonic functions (spectral method), some of the 2D dynamo computations were carried out by finite difference method, see for example (*Braginsky, 1978, Braginsky and Roberts, 1987, Cupal and Hejda, 1989, Anufriev et al., 1995, Anufriev and Hejda, 1998*).

Our new numerical method deals directly with three components of magnetic and velocity fields (in the spherical system of coordinates). The components are resolved in the physical space for r- and θ-coordinates and expanded in the sin- and cos- series for φ-coordinate. Development of this algorithm was aimed at finding a method, which would better fit for the inviscid hydromagnetic dynamos. From the numerical point of view, the inviscid models (i.e. models in which the viscosity is neglected in the bulk of the core) are characterized by the fact that the sharp changes of velocities in the boundary layers are replaced by jumps and the velocity in the bulk of the core is expressed in

1 Address: Boční II, 141 31 Prague 4, Czech Republic (e-mail: ph@ig.cas.cz)

2 Address: B. Gruzinskaya 10, Moscow, Russia, 123810 (e-mail: rm@uipe.srcc.msu.su)

the form of integrals in z direction (for details see for example *Anufriev, 1995*). The solution thus requires a transition between spherical and cylindrical coordinates at each time step. This can be more simply accomplished by grid method then by spectral one. A similar treatment was used by *Nakajima and Roberts (1995)* who solved the problem in cylindical system of coordinates and avoided the transition between the systems by a special mapping method.

Our new method was explained in (*Hejda and Reshetnyak, 1999*), hereinafter denoted as HR. Now we extend this approach to the models with solid inner core and we take into consideration also the effects caused by jumps of the velocities across the boundaries. We report tests for free decay modes. Then we describe a simple model with different magnetic diffusivity η at the outer and inner core. And at last, we demonstrate some results of numerical simulations with imposed velocity field similar to *Gubbins' model (1973)* (see also *Dudley & James, 1989*), but with the inner core.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

The kinematic dynamo problem consists in solution of the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\nabla \times \mathbf{B}) - \nabla \times \eta \nabla \times \mathbf{B} \ , \tag{1}$$

for the non-divergent

$$\nabla \cdot \mathbf{B} = 0 \ , \tag{2}$$

magnetic field, where $\mathbf{V}$ is an imposed velocity field usually treated as incompressible ($\nabla \cdot \mathbf{V} = 0$).

As was explained above, the velocity changes abruptly at the inner-outer core boundary (IOCB) and core-mantle boundary (CMB). The magnetic diffusivity can have a jump at IOCB but is constant in either region. The mantle is supposed to be non-conductive. Analyzing (1) and (2) in the thin boundary layer of thickness δ and taking into account that (a) typical time changes in the layer are much smaller than those in the bulk of the core, i.e. l.h.s. of (1) can be neglected, (b) changes of magnetic field and velocity along the layer are neglegible in comparisson with the changes across the layer and (c) all jumps that are $O(\delta)$ can be neglected, we come to following results: Magnetic field and derivatives of its radial component are continuous in the whole space. The radial derivative of tangential components has on IOCB the jump

$$\left[\eta \frac{\partial}{\partial r} \mathbf{B}_{\mathcal{T}}\right] = -B_r [\mathbf{V}_{\mathcal{T}}] \tag{3}$$

and $\mathbf{B}$ links on the CMB to the potential source-free magnetic field with the jump

$$\left[\frac{\partial}{\partial r} \mathbf{B}_{\mathcal{T}}\right] = -B_r [\mathbf{V}_{\mathcal{T}}] \ . \tag{4}$$

Following HR we present the solution in the form

$$\mathbf{b}(r,\theta,\varphi,t)=\begin{pmatrix} b_r \\ b_\theta \\ b_\varphi \end{pmatrix}=\sum_{m=0}^{M}\begin{pmatrix} b^c_{r_m} \\ b^c_{\theta_m} \\ b^c_{\varphi_m} \end{pmatrix}\cos m\varphi+\begin{pmatrix} b^s_{r_m} \\ b^s_{\theta_m} \\ b^s_{\varphi_m} \end{pmatrix}\sin m\varphi \ , \tag{5}$$

where $\mathbf{b}=r\mathbf{B}$ and (r,θ,φ) is a spherical system of coordinates. Introduction of a new variable $\mathbf{b}$ instead of $\mathbf{B}$ allow us to deal properly with the singularity of the magnetic field in the centre of the sphere.

Since spherical coordinates are being used, one must formulate the boundary conditions not only on the surface of the sphere, but also at the centre ($r=0$) and along the axis of rotation ($\theta=0$, $\theta=\pi$): To do so we can make use of the fact that the free decay modes form a basis of the space in which the solution is found. We get following conditions: in the centre ($r=0$) all components of $\mathbf{b}$ are zero; at the axis of rotation ($\theta=0$ or $\theta=\pi$) $b_{rm}=0$ if $m\neq 0$ and $\dfrac{\partial b_{rm}}{\partial\theta}=0$ if $m=0$; $b_{\theta m}=b_{\varphi m}=0$ if $m\neq 1$ and $\dfrac{\partial b_{\theta m}}{\partial\theta}=\dfrac{\partial b_{\varphi m}}{\partial\theta}=0$ if $m=1$. For details see HR.

In contrast to the toroidal-poloidal decomposition, where the non-divergence is automatically satisfied, in this case we must take deal with it. At the first glance, the situation seems to be rather curious, as there are four equations (1) and (2) for just three components of magnetic field $\mathbf{B}$. Nevertheless, it only appears so. The core of a subject is that if the initial vector $\mathbf{B}_{init}$ is non-divergent, the exact solution of (1) preserves this property. In the case of the numerical solution there is the danger that the non-divergence of the solution will deteriorate due to numerical errors. We, therefore, integrate Eq. (1) only for θ and φ components and compute the r component from condition (2).

The substitution of (5) into Eqs (1) and (2) leads to the following system:

$$b^{c,s}_{r_m}=-\frac{1}{r\sin\theta}\int_0^r\left(\frac{\partial}{\partial\theta}\left(\sin\theta b^{c,s}_{\theta_m}\right)\pm m b^{s,c}_{\varphi_m}\right)dr \ , \tag{6a}$$

$$\frac{\partial b^{c,s}_{\theta_m}}{\partial t}=\nabla_1^2 b^{c,s}_{\theta_m}-\frac{1}{r^2}\frac{m^2}{\sin^2\theta}b^{c,s}_{\theta_m}-\frac{b^{c,s}_{\theta_m}}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial b^{c,s}_{r_m}}{\partial\theta}\mp\frac{2m}{r^2}\frac{\cos\theta}{\sin^2\theta}b^{s,c}_{\varphi_m}+R^{c,s}_{\theta_m} \ , \tag{6b}$$

$$\frac{\partial b^{c}_{\varphi_m}}{\partial t}=\nabla_1^2 b^{c,s}_{\varphi_m}-\frac{1}{r^2}\frac{m^2}{\sin^2\theta}b^{c,s}_{\varphi_m}-\frac{b^{c,s}_{\varphi_m}}{r^2\sin^2\theta}\pm\frac{2m}{r^2\sin\theta}b^{s,c}_{r_m}\pm\frac{2m\cos\theta}{r^2\sin^2\theta}b^{s,c}_{\theta_m}+R^{c,s}_{\varphi_m} , \tag{6c}$$

where

$$\nabla_1^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \tag{7}$$

and the convective term $\mathbf{R}$:

and the convective term **R**:

$$\mathbf{R} = r(\nabla\times(\mathbf{V}\times\mathbf{b}/r)) = (\mathbf{b}\cdot\nabla)\,\mathbf{V} - r(\mathbf{V}\cdot\nabla)(\mathbf{b}/r) \ . \tag{8}$$

The rules for decomposing of **R** into sine and cosine Fourier series are described in HR. For better understanding the process as well as to check the stability and correctnes of our solution we compute the balance of magnetic energy. Taking the scalar product of (1) with **B** and integrating over the bulk of the core we get

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{R}^3} B^2 d\mathbf{r}^3 = \int_{\mathcal{R}^3}[\mathbf{V}\times\mathbf{B}]\cdot[\nabla\times\mathbf{B}]d\mathbf{r}^3 - \eta\int_{\mathcal{R}^3}(\nabla\times\mathbf{B})^2\,d\mathbf{r}^3 - \int_{\mathcal{S}} V_r B^2\,d\mathcal{S} +$$

$$+\int_{\mathcal{S}} B_r(\mathbf{B}\cdot\mathbf{V})\,d\mathcal{S} - \eta\int_{\mathcal{S}}[\nabla\times\mathbf{B}]\times\mathbf{B}\,d\mathbf{S} \ . \tag{9}$$

Following *Anufriev et al. (1993)* we introduce the function of imbalance which is equal to the difference of the r.h.s. and l.h.s. of Eq. (9). This quantity can be used to control the parameters of the numerical process, e.g., to optimize the length of the time-step.

3. NUMERICAL RESULTS

The finite difference numerical scheme with central difference approximation of the 2nd order of accuracy was applied on grid $\mathcal{G} = \{(r_i,\ \theta_j),\ i = 1,\ldots,\ N,\ j = 1,\ldots,\ K\}$, where $\theta_j = (j-1)h_\theta$, $h_\theta = \pi/(K-1)$ and the grid size in r may be regular or irregular. The diffusions (second-order derivative terms) in Eqs (6b, 6c) were carried out implicitly, whereas the terms with the first-order derivatives were treated explicitly. The corresponding systems of linear equations were solved by the Gauss-Seidel method.

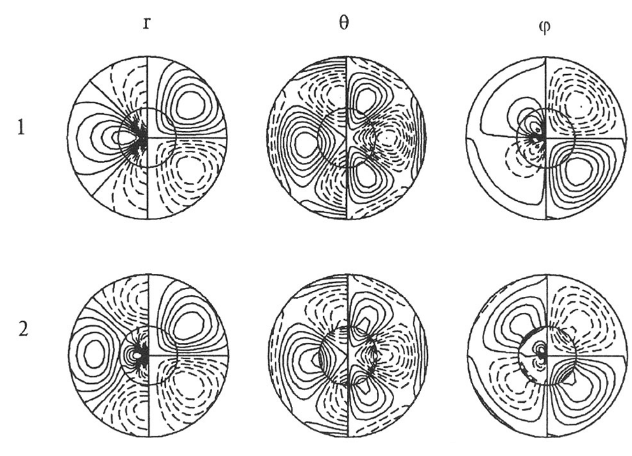

Fig. 1. Maps of the free decay mode P_3^2 with $\eta_1 = \eta_2$ (row 1) and $\eta_1 = 0.3\eta_2$ (row 2); r-, θ- and φ- spherical components are shown in the meridional projection for $\varphi = 0$ (left side of the circles) and in the equatorial plane (right side of the circles). The numbers denote the minimal and maximal values.

We checked our model by free decay modes (up to order 6) with equal magnetic diffusivity η in the inner and outer core and obtained $\sim N^2 (K^2)$ convergence to the analytical solution. For details see HR. In order to check how the solution depends on the inner core conductivity, calculations were carried out with higher value of the latter (Fig. 1). According to (3), the r-component is smooth across IOCB, whereas the other components suffers with a jump. Let us notice that the r and φ components are symmetrical with respect to the equator. The non-zero values on the equator shown in Fig. 1 are just errors of the numerical approximation.

At last, we have made comparison with the kinematic dynamo of Gubbins. The model was originally defined by velocity field

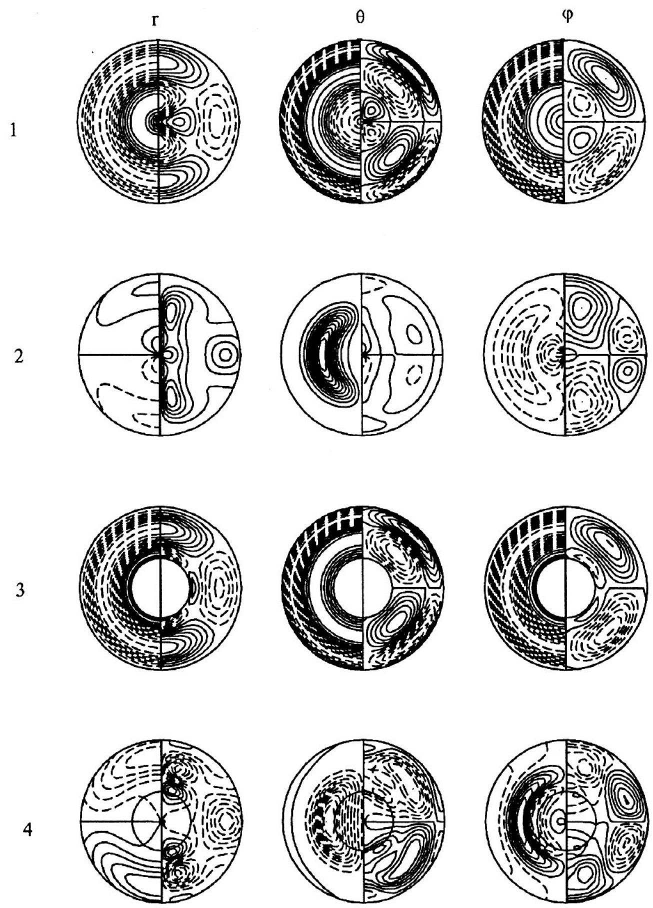

Fig. 2. The maps of the velocity (rows 1, 3) and magnetic fields (2, 4) for the Gubbins model without (1, 2) and with (3, 4) the inner core for $R_m = 53$, $\eta_1 = \eta_2$; r-, θ- and φ- spherical components are shown in the meridional projection for $\varphi = 0$ (left side of the circles) and in the equatorial plane (right side of the circles).

$$\mathbf{V} = R_m\left(\mathbf{T}_2^0 + \varepsilon\mathbf{S}_2^0\right) , \tag{8}$$

($\varepsilon = 0.1$) and

$$T_2^0(r) = S_2^0(r) = -r^2 \sin(2\pi r) \tanh(2\pi(1-r)) \tag{9}$$

(*Gubbins, 1973*). The steady (stationary) states $R_m = 54.7$ and -96.7 were accurately determined by *Dudley and James (1989)* and these values were confirmed by HR. In the recent paper we have used the same velocity but with $\mathbf{V} = 0$ in the inner core (for $r \le 0.35$). In contrast to the former case, the solution is now oscillatory, which makes a direct comparison rather difficult. Nevertheless, the effect of the jump of velocity on the tangential components of B is evident (Fig. 2).

4. CONCLUSIONS

One of the main problems in solving the geodynamo is to check the correctness of the results. Since, in the general case, the equations have no analytical solutions the best way is to compare results obtained by various methods. As our method differs substantionally from the traditional spectral approach, the good coincidence with previous results is significant. In order to avoid the mistakes it is also important to divide the computer code into several steps end check carefully every step. In harmony with this principles the project of 3D inviscid geodynamo modelling have been started with the solution of induction equation. In the first step (see HR) the computer code of the model without the inner core was developed and tested by free decay modes and by comparison with kinematic dynamo calculations of *Dudley and James (1989)*.

The method have been now enlarged to the case with the solid inner core. The jumps of velocity and conductivity at the boundaries of the liquid core were also taken into account. The computer code thus satisfies all demands put on the solution of induction equation in the frame of the 3D inviscid geodynamo models.

Acknowledgments: RM is grateful to Geophysical Institute of the Academy of Science of the Czech Republic in Prague for the support of his stay. The research was supported by the grant of RFBR 97-05-64797 and by the grant 205/97/0897 of the Grant Agency of the Czech Republic.

Manuscript received: 16 June 1999; Revisions accepted: 7 July 1999

References

Anufriev A.P., Cupal I. and Hejda P., 1993: Time evolution of the solution in model Z. The Cosmic Dynamo., *IAU Symp Series*, **157**, 447-452.

Anufriev A.P., 1995: The influence of solid core on the Earth's hydrodynamics. *Geophys. Astrophys. Fluid. Dyn.*, **77**, 15-25.

Anufriev A.P., Cupal I. and Hejda P., 1995: The weak Taylor state in an $\alpha - omega$-dynamo. *Geophys. Astrophys. Fluid Dyn.*, **79**, 125-145.

Anufriev A.P. and Hejda P. 1998: Effect of the magnetic field at the inner core boundary on the flow in the Earth's core. *Phys. Earth Planet. Inter.*, **106**, 19-30.

Braginsky S.I., 1978: Nearly axially symmetric model of the hydromagnetic dynamo of the Earth. *Geomag. and Aeronom.*, **18**, 225-231.

Braginsky S.I. and Roberts P.H., 1987: A model-Z geodynamo. *Geophys. Astrophys. Fluid Dyn.*, **38**, 327-349.

Bullard E.C. and Gellman H., 1954: Homogeneous dynamos and terresrtial magnetism. *Phil. Trans. R. Soc. Lond.*, **A 247**, 213-278.

Cupal I. and Hejda P., 1989: On the computation of a model-Z with electromagnetic core-mantle coupling. *Geophys. Astrophys. Fluid Dyn.*, **49**, 161-172.

Dudley M.L. and James R.W., 1989: Time dependent kinematic dynamos with stationary flows. *Proc. R. Soc. Lond.*, **A 425**, 407-429.

Glatzmaier G.A. and Roberts P.H., 1995: A three-dimension self-consistent computer simulation of a geomagnetic field reversal. *Nature*, **377**, 203-209.

Gubbins D., 1973: Numerical simulations of the kinematic problem. *Proc. Trans. R. Soc. Lond.*, **A 274**, 493-521.

Hejda P. and Reshetnyak M., 1999: The grid-spectral approach to 3D-geodynamo modelling. *Computers and Geosciences* (in press).

Kuang W. and Bloxham J., 1997: An Earth-like numerical dynamo model. *Nature*, **389**, 371-374.

Nakajima T. and Roberts P.H., 1995: An application of mapping method to asymmetric kinematic dynamos. *Phys. Earth Planet. Inter.*, **91** , 53-61.