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Combined grid-shell approach for convection in a rotating spherical layer

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Abstract. – A model of Boussinesq convection in a fast rotating spherical layer with the free-rotating inner core is considered. To solve the problem, two sets of equations are used. First, we solve the original Navier-Stokes equation and thermal-flux equation on the coarse grid. The small-scale flow features are described in terms of the shell model technique. The influence of small scales on large-scales is taken into account by means of effective viscosity, which is calculated from the spectral energy flux defined in the shell model. The properties of the turbulent spectrum is considered.

Introduction. – Astrophysical and geophysical convection (as well as convection encountered in engineering processes) usually involves a very wide range of scales, so that a direct numerical simulation of the problem requires usage of various simplifying assumptions. For example, the geodynamo problem considers flows with the Reynolds number up to $Re \approx 10^9$ (*e.g.*, [1]) and a comprehensive direct numerical simulation will require a grid with about $N \approx Re^{9/4} \approx 10^{20}$ nodes. One way to make the problem acceptable for available computer facilities is to rescale the dimensionless numbers, reasoning from the idea of turbulent transport coefficients and neglecting small-scale motion [2]. In this case, the turbulent transport coefficients are estimated from various semiempirical models of turbulence [3].

Here we suggest a further step and describe the subgrid turbulence in the framework of the shell model. The shell model is a finite-dimensional model of turbulent motions, yet it gives a correct description of their spectral properties. In many cases it provides us with a comprehensive information concerning small-scale turbulence we are interested in. The shell model requires reasonable computer facilities to ensure numerical realization of the proposed approach.

We apply this general approach to the convection in a spherical layer heated from below. This choice of geometry comes from the particular interests to the geodynamo (or astrophysical dynamo) problem. Here, basing on the assumption of the homogeneous and isotropic small-scale turbulence, we consider only one shell model for the whole spherical layer. This simplification can be relevant on the small scales of real objects only, and should be considered as a first step in this approach. We also discuss possible applications of this technique to the case with the magnetic field.

The large-scale equations. – We have developed our approach to facilitate the treatment of the convection problem in a rotating spherical layer which is important for understanding the Earth and stellar convection. Convection of incompressible fluid ($\nabla \cdot \boldsymbol{V} = 0$) in the Boussinesq approximation in the layer ($r_i < r < r_0$), rotating with the angular velocity Ω, is described by the Navier-Stokes equation

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\, \boldsymbol{V} = -\nabla P + \boldsymbol{F} + P_r \nabla^2 \boldsymbol{V} \tag{1}$$

and the heat flux equation

$$\frac{\partial T}{\partial t} + \boldsymbol{V} \cdot \nabla (T + T_0) = \nabla^2 T. \tag{2}$$

where $r_i = 0.35$, $r_0 = 1$ are similar to that of the Earth; $\boldsymbol{V}$ and T are the velocity field and temperature deviation from the prescribed temperature profile $T_0 = \frac{r_i/r-1}{1-r_i}$ [4]. Velocity, pressure and time are measured in terms of κ/L, $\rho\kappa^2/L^2$ and L^2/κ, respectively, where $L = r_0$ and κ is the thermal diffusivity, $P_r = \nu/\kappa$ is the Prandtl number, ν is the kinematic viscosity. The force F includes the Coriolis and Archimedean effects:

$$\boldsymbol{F} = -R_o^{-1}(\mathbf{1}_z \times \boldsymbol{V}) + R_a P_r \mathrm{Tr} \mathbf{1}_r, \tag{3}$$

where (r, θ, φ) is the spherical coordinate system, and $\mathbf{1}_z$ is the unit vector along the axis of rotation. $R_o = \kappa/(2\Omega L^2)$ is the modified Rossby number and $R_a = \alpha g_0 \delta T L^3/\kappa\nu$ is the Rayleigh number, α is the coefficient of volume expansion, δT is the drop of temperature through the layer and g_0 is the gravity acceleration at $r = r_0$. Equations (1)-(3) are closed by the non-penetrating and no-slip boundary conditions for velocity field and zero boundary conditions for temperature perturbations. Equations (1)-(3) can describe the large-scale convection properly, provided that the grid mesh does not exceed the scale of the obtained solution.

The problem arises when the grid mesh is larger than the typical viscous scale $\lambda \sim L Re^{-3/4}$ (see, *e.g.*, [5]). Here $Re = UL/\nu$ is the Reynolds number, which in our case can be related to the Rayleigh number as $Re = (R_a/P_r)^{1/2}$ (the characteristic velocity U is estimated from the relation $\rho U^2 \approx \rho\alpha\delta T g L$). There are two ways to solve the problem: the most primitive is to increase the number of points, which itself is rather problematic for 3D problems with $Re \gg 10^3$. The second way is to describe the turbulent effects using some approximate description for the small-scale variables. For this purpose we apply ourselves to the so-called shell models of turbulence.

The shell models. – The shell models were introduced in the seventies [6, 7] as an attempt to mimic the Navier-Stokes equations via dynamical systems with limited degrees of freedom. They are constructed by truncation of the Navier-Stokes equations in the Fourier space, retaining only one real or complex mode U_n as a representative of all modes in the shell with a wave number k ranging from $k_n = k_0\lambda^n$ to $k_{n+1} = k_0\lambda^{n+1}$. The parameter λ characterizes the ratio between two adjacent scales. It is one of the parameters of the model, which is usually taken equal to $\lambda = 2$ (then every shell corresponds to an octave of the wave numbers). Hereinafter, we shall use this value. The coupling between the shells is chosen such as to preserve the main symmetries and properties of the Navier-Stokes equations. In spite of the obvious fact that the shell models give only a simplified description of turbulence, they prove to be a reasonable tool in the turbulent studies. For an introduction to shell models the readers are referred to [8].

In this paper, we take for a basis the so-called GOY shell model (Gledzer [7], Ohkitani and Yamada [9]). The properties of this model were investigated in detail by Biferale *et al.* [10], and Frick *et al.* [11]. Since we intend here to solve a shell model together with eqs. (1)-(3) we wrote the usual GOY equation in terms of real variables U_n. Here we assume that no force is essential on small scales, described by the shell equations. It means that the Bolgiano scale, which characterizes the smallest turbulent scale, affected by the Archimedean force, is larger than the grid mesh. We also omitted the Coriolis term, having in mind that the Coriolis force does not input any energy into the system. (Note that for the full anysotropical model its effect can be important, see [12].) Then we can write

$$d_t U_n = k_n \left(U_{n+1}U_{n+2} - \frac{1}{4}U_{n-1}U_{n+1} - \frac{1}{8}U_{n-2}U_{n-1} \right) - P_r k_n^2 U_n. \tag{4}$$

In the inviscid limit this system of shell equations is characterized by two conservation laws. The first conserved quantity E corresponds to the energy and the second one H can be considered as an analogue to helicity:

$$E = \sum_n |U_n|^2, \qquad H = \sum_n (-1)^n k_n |U_n|^2. \tag{5}$$

Equations (4) are integrated for $n = n_{\min} \dots n_{\max}$. The left limit is determined by the grid mesh d ($k_{n_{\min}} = 2\pi/d$), the right one must enable one to resolve the smallest scale, which can be excited for a given Rayleigh number.

Combined model. – A more delicate point is to provide a correct linkage between the large-scale (grid) and small-scale (shell) simulations. Our main goal at the first stage is the evaluation of the energy dissipation on small (subgrid) scales and corresponding energy outflow from large scales. To do this we must first prescribe at each time instant the value of the variable U at two adjacent points, $U_{n_{\min}}$ and $U_{n_{\min}+1}$, which reflect the instantaneous intensity of velocity oscillations on the typical scale resolved by the grid. The shell energy $E_n = U_n^2$ corresponds to the energy contained in the whole octave of the wave numbers $U_n^2 = \int_{k_n}^{k_{n+1}} |\hat{V}(\vec{k})|^2 \mathrm{d}\vec{k} \approx k_n |\delta V_n|^2$. So far the large-scale equations are based on the finite-difference approximation, while the estimates of $U_{n_{\min}}$ and $U_{n_{\min}+1}$ use the second-order structure function $S_2^F(l) = \langle |F(x+l) - F(x)|^2 \rangle$ for any field F calculated for arbitrary non-homogeneous grid mesh, where $\langle \dots \rangle$ means averaging over the whole set of pairs of mesh points with $k_n < l^{-1} < k_{n+1}$.

The second essential point in the interaction of these two models is the influence of the small-scale model on large scales. This effect can be described by introducing the effective (turbulent) viscosity ν^T and corresponding corrections for the diffusion coefficients in the large-scale model. According to Kolmogorov's assumptions, the dissipation scale λ can be derived from the dissipation rate ε and molecular viscosity ν as $\lambda \approx (\nu^3 \varepsilon^{-1})^{1/4}$. Then the effective turbulent viscosity, which provides the same dissipation rate but on the grid scale, can be estimated as $\nu^T = 0.1(\delta^4 \varepsilon)^{1/3}$. Here the numerical coefficient is defined from the comparison of our formulas with Smagorinsky's model [13]. The dissipation rate ε is calculated as the total energy dissipated in the shell model per time unit $\varepsilon = \sum_{n=n_{\min}+2}^{n_{\max}} (k_n U_n)^2$.

The last assumption concerns the large-scale viscosity $\hat{\nu} = \nu + \nu^T$ and corresponding diffusion terms. We suppose that the diffusion terms in eqs. (1)-(2) can be rewritten as $P_r(1+\nu^T)\nabla^2 \boldsymbol{V}$ and $(1+P_r\nu^T)\nabla^2 T$, which is equivalent to the assumption that the turbulent Prandtl number in the regime of developed turbulence tends to unity. Note that the detailed description supposes the separated shell model for each grid node of the large-scale model. Here we make only a mean over the whole volume estimate of turbulent characteristics, using only one shell model.

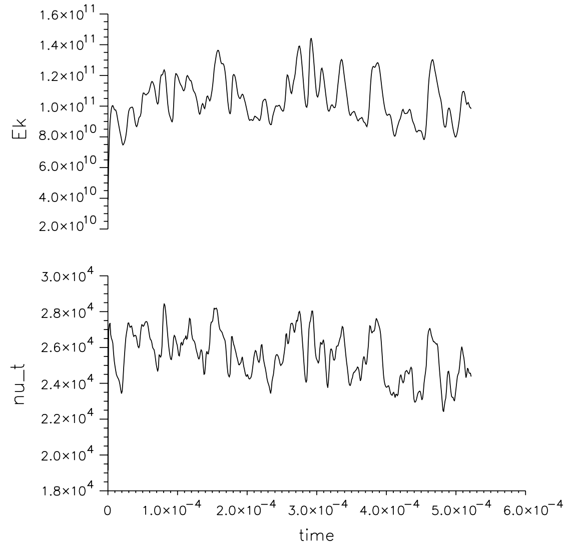

Fig. 1 – Evolution of the kinetic energy (upper) and variations of the subgrid viscosity (lower) plots.

To solve the large-scale equations (1)-(3) we used the control volume method described in details in [14]. The typical grid mesh was $(r, \theta, \varphi) = (16, 16, 16)$. These values give the estimations only because we use a staggered grid technique and the number of points for different physical fields and meshes is different. Because the spherical homogeneous grid leads to non-homogeneous distances between the points, the estimate of the minimal resolved scale d is not trivial. Here and in the following we use the upper estimate $d = 1/16$. Note that in the vicinity of the inner core and poles this scale is more than one order smaller. Following these estimations we choose $n_{\min} = 4$. The maximum number of shells follows from the estimate of Kolmogorov's diffusion scale: $k_{\mathrm{n_{max}}} > \left(\frac{R_a}{P_r}\right)^{3/8}$. In our case we used $n_{\max} = 19$ (however, this number can be enlarged if required).

It should be noted that the turbulent diffusivity as well as the mean velocity are treated as the values averaged over the timescale of turnover time on the spatial scale of matching of the shells and the grid. The turnover time for Kolmogorov's scale $\tau_\lambda = \left(\frac{R_a}{P_r}\right)^{-3/4}$ defines the time step τ_{s} for integration of the shell equation. The time step τ_{g} for the grid equations was defined from the requirement of fulfillment of conservation laws in the control volume (see [14] for details). Obviously, $\tau_{\mathrm{s}} \ll \tau_{\mathrm{g}}$, in our case their ratio was 1000. The shell models have a highly irregular oscillatory solution, which can be used after averaging over time. Here for averaging time we use the turnover time $\tau_l = \tau_\lambda \left(\frac{d}{L}\right)^{2/3} \left(\frac{R_a}{P_r}\right)^{-1/2}$, where d is a grid scale. In our case, we put $d \approx L$ and $\tau_l = 1000\tau_\lambda$. The same turnover time was used for averaging the grid data to determine $U_{\mathrm{n_{min}}}$, $U_{\mathrm{n_{min}}+1}$ in the shell model.

The results of simulations. – Simulations were made on the single processor Pentium and took about some hours. Figure 1 shows the time evolution of the kinetic energy of supergrid scales obtained by the method of shell model. This figure also shows the variation of the subgrid viscosity ν^T. Note that simulation without the subgrid viscosity leads to an essential growth of kinetic energy and noisy energy variations. A smooth behavior of kinetic energy is provided by variations of the subgrid viscosity (the lowest curve in the same figure).

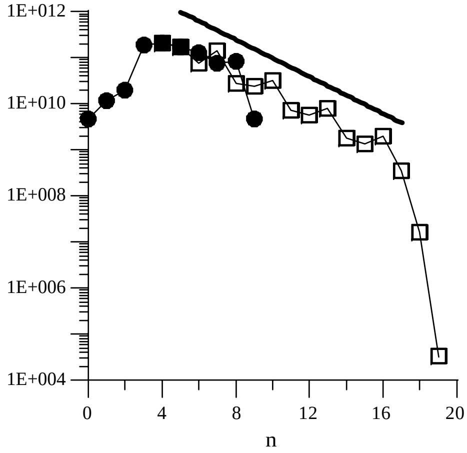

Fig. 2 – Structure function S_2 (circles) calculated from the grids and shell spectrum (squares) for $R_a = 3 \cdot 10^{12}$, $R_o = 10^{-4}$, $P_r = 0.1$. The straight line indicates the slope of $k^{-2/3}$, which corresponds to Kolmogorov's spectrum.

Figure 2 shows spectral characteristics for both super- and subgrid ranges of scales. The large (supergrid) scales are described by the second-order structure function S_2 (black circles). The small (subgrid) scales are characterized by the mean shell energy $\langle U_n^2 \rangle$ indicated by black squares. The conjugation between the grid and shells occurs at $n_{\min} = 4$. The tail of the structure function ($n = 5$–8) corresponds to polar regions, where the spherical grid has better resolution. Note that the spectra coincide up to $n = 7$. An essential kinetic energy dissipation occurs in shells with $n > 12$. The shell with $n = 12$ is responsible for the scales two order of magnitude lower than the typical grid scale. The slope of the curves for $6 \le n \le 15$ is "$-2/3$", which corresponds to Kolmogorov's spectrum "$-5/3$".

To summarize, the approach developed in this paper allows us to resolve the small-scale fields in the wide range of subgrid scales and use the obtained information in the large-scale simulations. The obtained Reynolds number is already of the same order as the expected one in the Earth. Our model makes it possible to describe the small-scale motions over 5 decades of scales. Nevertheless, we consider the possibility to extend this range of scales when more effects in the shell model are taken into account and spectra are flatter. A natural development of the method suggested can be based on the usage of a family of shell models to reproduce spatial variability of turbulence properties.

We also realize that the magnetic field is the crucial point in the modern models of the Earth's core and our approach must include this field. So far, the shell model technique has already been developed for MHD turbulence (see, *e.g.*, the paper by Frick and Sokoloff [15], who introduce a MHD shell model which provides the conservation of MHD conservation laws; temperature fluctuation can also be included into the shell model), the main expected problem here is the matching of grids and shell models. We consider this as our main task in the future.

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