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GEOPHYSICS

Cascade Models of Turbulence for the Earth's Liquid Core

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At present, the liquid core of the Earth represents an object of intense interdisciplinary investigation. Since this part of the Earth is responsible for the generation of the planetary magnetic field, it is no wonder that several models of magnetic hydrodynamic (MHD) convection in the liquid core were proposed within recent years ([1] and references therein). Three-dimensional dynamo models that are known to date and based on mechanisms of thermal (concentration) convection (see, for example [2–4]) make it possible to reproduce several observed features of the geomagnetic field, such as spatial spectrum of the geomagnetic field, geomagnetic field inversions, and prevalence of magnetic energy over the kinetic energy of convective motions. Estimates of the velocity and direction of rotation of the Earth's solid core (with respect to the mantle), based on models including the mechanism of solid core rotation, are compatible with seismological data [5]. The problem of small values of diffusion coefficients is one of the main difficulties encountered in the simulation of MHD processes in the Earth's liquid core. In other words, information on a wide range of spatiotemporal scales is necessary for describing processes in the liquid core. The situation with hydrodynamics is the most critical.

For a liquid core with radius L = 3500 km, evaluation of the Reynolds number $\mathrm{Re}^M = \frac{LV_{wd}}{\nu^M}$ by the western drift rate of the geomagnetic field $V_{wd} \approx 0.2°\ \mathrm{yr}^{-1}$ yields $\mathrm{Re}^M \sim 10^9$, where the kinematic viscosity coefficient $\nu^M = 10^{-6}\ \mathrm{m^2\,s^{-1}}$ [6] (hereafter, index M indicates that characteristic numbers were calculated using the molecular values of diffusion coefficients). Such a large value of the Reynolds number matches the state of developed turbulence. Simple evaluations of the number of degrees of freedom for the Kolmogorov turbulence [7] in the 3D problem yield $N = \mathrm{Re}^{9/4} \sim 10^{20}$; i.e., 10^{20} grid points are necessary for a discrete description of the problem. Note that the most detailed present-day models of the geodynamo [2, 3] provide an average spatial resolution of $\sim(10^{-2}–10^{-3})\ L$, which is still far from the required $10^{-9}\ L$. A similar evaluation of the Peclet number $\mathrm{Pe}^M = \frac{LV_{wd}}{\kappa^M} \sim 10^8$, where the coefficient of molecular thermal conductivity $\kappa^M = 10^{-5}\ \mathrm{m^2\,s^{-1}}$, also points to the necessity for resolving a large number of scales. Finally, although evaluation of the magnetic Reynolds number $\mathrm{R}_m^M = \frac{LV_{wd}}{\eta^M} \sim 10^3$ is the lowest of all the above-mentioned estimates, it also requires the use of the most modern computers. The foregoing evaluations of dimensionless parameters Re^M, Pe^M, and R_m^M confirm the statement that convection in the Earth's liquid core is turbulent, and special approaches are needed for its description.

The available sufficiently wide range of semiempirical models of turbulence (see the review in [8]) allow us to assess energy dissipation on small scales. The application of such models is formally reduced to the computation of effective coefficients of viscosity and their use in large-scale models. Models of this type are extensively employed in technology. However, until recently, they were not adapted to the case of strong magnetic fields. Let us recall here that magnetic energy concentrated within the Earth's core exceeds the kinetic energy (in the mantle-related reference system) by several orders of magnitude. Semiempirical models based on the integral characteristics of small-scale fields do not allow us to keep track of spectral properties. In the present work, we propose to make up for this drawback with the help of cascade models of turbulence.

Cascade models were proposed in the 1970s [9, 10] for simulating the Navier–Stokes equation behavior with the assistance of dynamic systems having a lim-

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ited number of degrees of freedom. The models were elaborated for variables that corresponded to field fluctuations with wave vector k in the range (shell) between $k_n = k_0\lambda^n$ and $k_{n+1} = k_0\lambda^{n+1}$. Parameter λ characterizes the ratio of two neighboring scales. Usually, $\lambda = 2$. Despite the fact that cascade models provide only a simplified description of turbulence, they adequately reproduce many of its properties [11, 12]. Based on separate developments of cascade models for the thermal convection in the Boussinesq approximation [13] and the MHD turbulence [14], we propose an analog of complete equations of the dynamo on the basis of thermal convection in terms of cascade models. Let us consider the following dimensionless system that consists of n ordinary differential equations and describes the evolution of velocity U_n, temperature Θ_n, and magnetic field B_n:

$$\mathrm{R}_0\frac{dU_n}{dt} = \mathrm{R}_0 k_n\left(-\frac{1}{8}U_{n-2}U_{n-1} - \frac{1}{4}U_{n-1}U_{n+1} + U_{n+1}U_{n+2}\right) + \mathrm{Ra}^M\gamma_2\Theta_n - k_n\left(\frac{1}{8}B_{n-2}B_{n-1} - \frac{1}{4}B_{n-1}B_{n+1} + B_{n+1}B_{n+2}\right) - \mathrm{E}^M k_n^2 U_n, \tag{1}$$

$$\frac{d\Theta_n}{dt} = k_n\left(U_{n+1}\Theta_{n+2} + U_{n-1}\Theta_{n+1} - \frac{1}{2}U_{n-2}\Theta_{n-1} + \Theta_{n+1}U_{n+2} - \frac{1}{2}\Theta_{n-1}U_{n+1} - \frac{1}{4}\Theta_{n-2}U_{n-1}\right) - \gamma_3 k_n^2\Theta_n,$$

$$\frac{dB_n}{dt} = -\frac{1}{6}k_n(U_{n-2}B_{n-1} - B_{n-2}U_{n-1} + U_{n-1}B_{n+1} - B_{n-1}U_{n+1} + U_{n+1}B_{n+2} - B_{n+1}U_{n+2}) - \gamma_1 k_n^2 B_n.$$

Here, time t and velocity U are measured in $\frac{L^2}{\eta}$ and $\frac{\eta}{L}$ units, respectively, where $\eta = 1$ m^2 s^{-1} is the value of the coefficient of molecular magnetic diffusion. The magnetic field is measured in $\sqrt{2\Omega\eta\rho\mu_0}$ units, where $\Omega = 7.29 \cdot 10^{-5}$ rad s^{-1} is the angular velocity of the Earth's daily rotation, $\rho = 10^4$ kg m^{-3} is the liquid core density, and $\mu_0 = 4\pi \cdot 10^{-7}$ H m^{-1} is the magnetic constant. Ratios of molecular diffusion coefficients to the η value are specified by the parameters $\gamma_1 = \frac{\eta^M}{\eta}$, $\gamma_2 = \frac{\nu^M}{\eta}$, $\gamma_3 = \frac{\kappa^M}{\eta}$. Dimensionless numbers are specified by the following relationships: the Rossby number $\mathrm{R}_0 = \frac{n}{2\Omega L^2}$, the Eckman number $\mathrm{E}^M = \frac{\nu^M}{2\Omega L^2}$, and the Rayleigh number $\mathrm{Ra} = \frac{\alpha g_0 \Delta T L}{2\Omega\kappa^M}$, where α is the coefficient

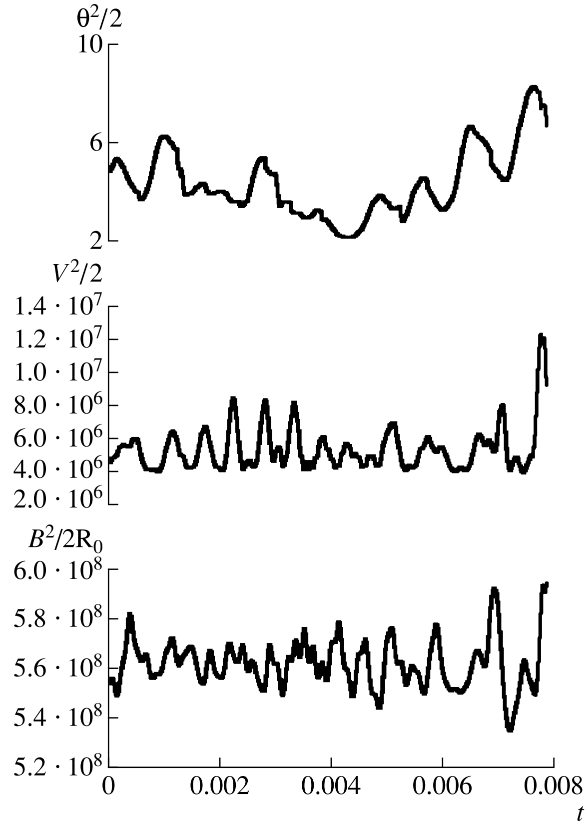

Fig. 1. Evolution of the squares of temperature, velocity, and magnetic field in the cascade model.

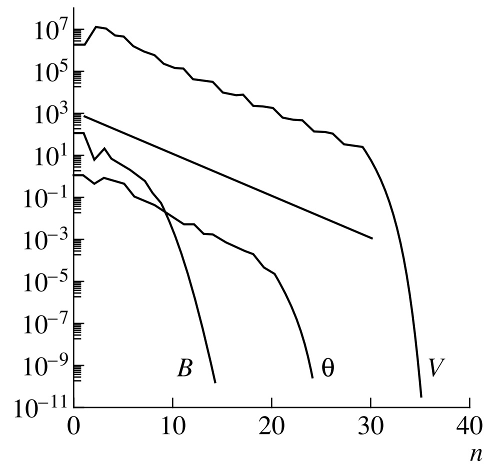

Fig. 2. Spectra of temperature θ, velocity *V*, and magnetic field *B*. Straight line corresponds to the Kolmogorov spectrum "$-\frac{5}{3}$".

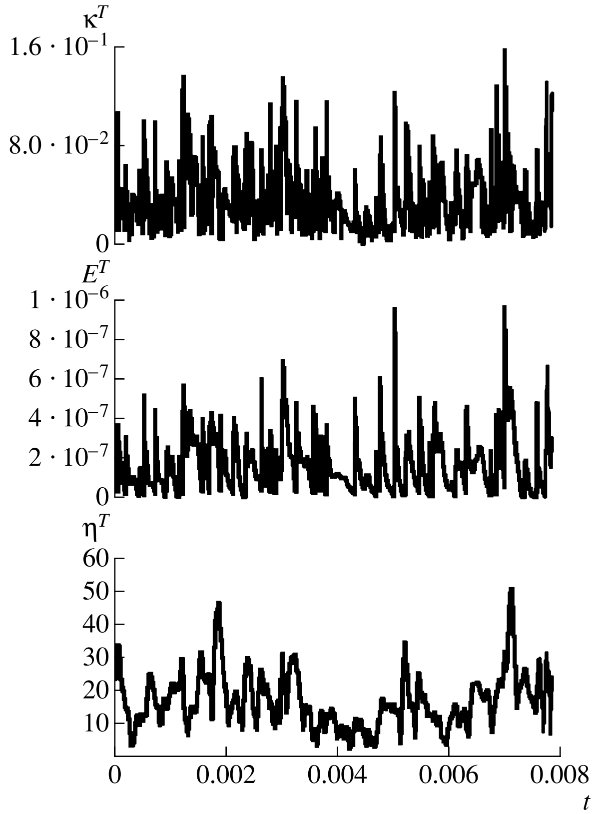

Fig. 3. Evolution of the coefficient of turbulent thermal conductivity κ^T, Eckman number E^T, and coefficient of turbulent magnetic diffusion η^T.

of volume expansion, g_0 is the free fall acceleration, and ΔT is the measure of temperature fluctuations. Our choice of measurement units makes it possible to obtain comparatively small dimensionless values of the fields, which is convenient for calculations. Note that the system of ordinary equations (1) is much simpler than the complete 3D system of MHD equations. However, it satisfies the basic conservation laws [13, 14] and allows us to use real physical parameters.

Equation system (1) can easily be solved on parallel computation systems. We used a cluster of three personal computers based on Alpha-21264 processors (each of the physical variables U, Θ, and B was calculated on a separate processor supported by an MPI 2.0).

Calculations were accomplished for the following parameter values: $E^M = 10^{-15}$, $R_0 = 4 \cdot 10^{-7}$, $Ra^M = 4 \cdot 10^7$, $\gamma_1 = 1$, $\gamma_2 = 10^{-6}$, and $\gamma_3 = 10^{-5}$. The last two values correspond to $\nu^M = 10^{-6}$ m^2 s^{-1} and $\kappa^M = 10^{-5}$ m^2 s^{-1}. Values $U_0 = V_{wd}$ and $B_0 = 10^{-2}$ T taken from observations [6] were used as the boundary conditions. In this case, the unit of time is $4 \cdot 10^5$ yr.

The temporal evolution of field squares ($\left(\frac{1}{2}\sum_n U_n^2,\ \frac{1}{2R_0}\sum_n B_n^2, \text{ and } \frac{1}{2}\sum_n \theta_n^2\right)$) is presented in Fig. 1. The pattern corresponds to a stationary quasi-periodical process with the magnetic energy exceeding the kinetic energy by two orders of magnitude. Spatial spectra of three fields are depicted in Fig. 2. Note that the widest range of scale variations presented in the velocity spectrum corresponds to the wave number $k_n = 2^{33} \approx 10^{10}$ or the resolution $10^{-10}L \approx 4$ mm, whereas the maximum scale in the model corresponds to the liquid core dimension L. The considered scale range completely covers both the inertial spectral region (corresponding to rectilinear segments of the spectra) and the dissipation region (sharp drop regions). Straight lines on the plot correspond to the Kolmogorov spectrum "$-\frac{5}{3}$".

In order to evaluate turbulent coefficients of viscosity, let us calculate the dissipation rate in the dimensional form for the fields U, Θ, and B as

$$\varepsilon_V = \nu^M \sum_n S(k_n U_n)^2, \quad \varepsilon_\Theta = \kappa^M C_p \sum_n S k_n^2 \Theta_n,$$
$$\varepsilon_B = \frac{\eta^M}{\rho\mu_0} \sum_n S(k_n B_n)^2, \tag{2}$$

where $S = 0.1$ is the Smagorinsky coefficient [15] and $C_p = 700$ J kg^{-1} K^{-1} is the thermal capacity of the medium [6]. Then the corresponding turbulent diffusion coefficient will be equal to $k_d^T = (\lambda^4\varepsilon)^{1/3}$, where λ corresponds to the minimal n used. In other words, if we consider a certain large-scale model with limited resolution $k_n \le k_{n_{\min}}$ ($\lambda \sim n_{\min}^{-1}$) and want to know the energy dissipation for large wave numbers $k_n > k_{n_{\min}}$, it is necessary to use evaluations (2). For the large-scale model in which turbulent values must be used instead of molecular diffusion coefficients, coefficients before the Laplacian operator will take the form

$$E^M \to E^M + E^M \gamma_2^{-2/3} B^2 f_V,$$
$$\gamma_3 \to \gamma_3 + \frac{(L^2 C_p \Delta T \gamma_3)^{1/3}}{\tilde{\eta}^{2/3}} f_T, \tag{3}$$
$$\gamma_1 \to \gamma_1 + R_0^{-1/3} \gamma_1^{1/3} B^2 f_B,$$

where multipliers f_V, f_Θ, and f_B correspond to ε_V, ε_Θ, and ε_B parts standing in (2) under the summation sign (ε_n) (in this case, all the values under the summation sign are dimensionless). The temporal behavior of E^T, κ^T, and η^T is illustrated in Fig. 3. It can be seen that the presence of turbulence cardinally alters the diffusion coefficient values. Evaluating the turbulent transport coefficients as $\nu^T = 10^2$, $\kappa^T = 10^{-2}$, and $\eta^T = 20$ m^2 s^{-1}, we obtain evaluations for the dimensionless parameters $Re^T \sim 10$, $Pe^T \sim 10^5$, and $R_m^T = 10^2$. These values can quite easily be used for the numerical simulation of large-scale processes of the dynamo. This approach makes it equally possible to combine cascade models with large-scale models when both models are simultaneously computed with mutual information exchange.

Let us mention some effects that are not taken into account in our model and may modify our results. We did not take into account the role of Coriolis forces, since they do not introduce any energy into the MHD system. However, they can in principle result in a certain suppression of turbulence owing to rotation and make it anisotropic. Our model is also limited by the following fact. Supposing that the spectrum subsequently acquires a dropping pattern, and selecting the Rayleigh number value in accordance with this condition, we do not calculate velocity and magnetic field values in the largest scale but instead adopt them from observations. Of course, a complete consistent model of the geodynamo, which includes the description of both large-scale variables in the grid model framework and small-scale variables in the cascade model framework, must abandon these limitations in the future.

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Erratum

In Fig. 3 of the paper "Monitoring of the Stressed-and-Strained State…" by V.A. Manuk'yan (*Doklady Earth Sciences*, 2002, vol. 384, no. 4, pp. 427–432), caption "refined cycle" should be replaced by "diurnal cycle." The author sincerely apologizes to the editorial board and readers.